DISCRETE STRUCTURE

Introduction

Discrete Mathematics deals with descrete objects. Discrete object are those objects that can be counted. For example:- Trees, integeres, houses, etc.

Logic

Logic is a language for resoning since logic can help us to reason the mathematical models. It need some rules associated with logic. So that we can apply those rules for mathematical resoning. Application maybe for designing circuits, programing, program verification, etc.

Propositions and Proportional Calculus

→ Proposition is a fundamental concept of logic.
→ Proposition is a declarative sencendence that is either true or false, but nott both.

e.g. 2+2=5 (false) ; 7-1=6 (true) ; Kathmandu is capital city of Nepal (true) are some of the examples of proposition.

x>5 ; come here ; 3+4 are some examples which are not propositions.

→ Propositions are denoted by using small letters like p, q, r, s... The truth value of proposition is denoted by T for true proposition and F for false proposition.

→ The logic that deals with proposition is called propositional logic or propositional calculus.

Logical Operators / Connectives

→ Logical operators are used to connect mathematical statements having one or more propositions by combining the propositions.

→ The combinal propositional is called compound proposition.

→ The truth table is used to get the relationship between truth values of propositions.

1. Negation (NOT)

Gives a proposition p, negation operator ($\neg$) is used to get negation of p, denoted by $\neg p$ called "not p".

Eg:- p = I love birds.
$\neg p$ = I don't love birds.

p	$\neg$P
T	F
F	T

2. Conjuction (AND)

Gives two propositions p and q, the proposition "p & q" denoted by $p \wedge q$ is the proposition that is true whenever p and q are true, false otherwise.

Eg:- p = "Ram is intelligent".
q = "Ram is diligent"
$p \wedge q$ = Ram is intelligent and diligent.

p	q	p∧q
T	T	T
T	F	F
F	T	F
F	F	F

3. Disjunction (OR):

Given two propositions p and q, the propositions "p OR q" denoted by p∨q is the proposition that is false whenever both the proposition p and q are false, true otherwise.

Eg:- p = "Ram is intelligent"
q = "Ram is deligent"
p∨q = "Ram is intelligent or deligent".

p	q	p∨q
T	T	T
T	F	T
F	T	T
F	F	F

Construct a truth table

1. $\neg p \wedge (p \vee \neg q)$

p	q	¬p	¬q	(p∨¬q)	¬p∧(p∨¬q)
T	T	F	F	T	F
T	F	F	T	T	F
F	T	T	F	F	F
F	F	T	T	T	T

ii. $(p \land q) \lor (\lnot q \land r)$

p	q	r	$\lnot q$	$p \land q$	$\lnot q \land r$	$(p \land q) \lor (\lnot q \land r)$
T	T	T	F	T	F	T
T	T	F	F	T	F	T
T	F	T	T	F	T	T
T	F	F	T	F	F	F
F	T	T	F	F	F	F
F	T	F	F	F	F	F
F	F	T	T	F	T	T
F	F	F	T	F	F	F

4. Exclusive Or (XOR)

Given two propositions p and q, the proposition exclusive or of p and q denoted by $p \oplus q$ is the propositions that is true whenever only one of the propositions p and q is true, false otherwise.

Eg:- p: "Ram drinks coffee in the morning"
q = "Ram drinks coffee in the morning".
$p \oplus q$ = Ram drinks tea or coffee in the morning

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

classmate

Date

Page

5. Implication

Given two propositional p and q, the proposition implication p→q, is the proposition that is false when p is true and q is false, true otherwise. Here p is called "Hypothesis" or "antecedent" or "premise" and q is called "conclusive" or "consequence".

Different terminologies to express p→q are like:-

1. "if p then q"
2. "q is consequence of p"
3. "p is sufficient of q"
4. "q if p"
5. "q is necessary for p"
6. "q follows from p"
7. "if p, q"
8. "p implies q"
9. "p only if q"
10. "q whenever p"
11. "q provides p"

p	q	p→q
T	T	T
T	F	F
F	T	T
F	F	T

Contrapositive, Inverse and Converse

Some of related implication formed from p→q are:-

Converse : $q \to p$

Inverse : $\neg p \to \neg q$

Contrapositive : $\neg q \to \neg p$

p = "today is sunday"
q = "it is not today"

implication = if today is sunday, it is hot.
Converse :- It is hot today only if today is sunday.
Inverse :- If today is not sunday, it is not hot.
Contrapositive :- If it is not hot today, it is not sunday.

Is contrapositive same as $p \to q$? Verify.

p	q	$p \to q$	$\neg p$	$\neg q$	$\neg q \to \neg p$
T	T	T	F	F	T
T	F	F	F	T	F
F	T	T	T	F	T
F	F	T	T	T	T

$$p \to q = \neg q \to \neg p$$

Q. Let p, q and r be the positions.

p = "You have the flu"
q = You ~~have~~ miss the final examination"
r = "You pass the course".

Express each of the proportion as an English Sentence and construct the truth table

① $p \to q$
If You have flue, you miss final examination.

(ii) $q \to \neg r$

If You miss the final exam, you will not pass the course.

(iii) $(p \to \neg r) \vee (q \to \neg r)$

If You have the flu, you will not pass the course or if you miss final exam, you will not pass the course.

p	q	r	$\neg r$	$p \to \neg r$	$q \to \neg r$	$p \to q$	$(p \to \neg r) \vee (q \to \neg r)$
T	T	T	F	F	F	T	F
T	T	F	T	T	T	T	T
T	F	T	F	F	T	F	T
T	F	F	T	T	T	F	T
F	T	T	F	T	F	T	T
F	T	F	T	T	T	T	T
F	F	T	F	T	T	T	T
F	F	F	T	T	T	T	T

Note:-

To translate english sentence to the proposition symbolic form, follows these steps:-

1. Restart the given sentence into building block sentence.
2. Give the symbol to each sentence and substitute the symbol using connectives:-

Q. If it is snowing then I will go to the beach.

$p \to$ "It is snowing" $q =$ 'I will go to beach"

$\Rightarrow p \to q$

q if p $p \rightarrow q$

p if q

p

Q. You can access the internet from campus, if you are a computer science major or you are not the freshman.

$\Rightarrow$

p = You can access the internet from campus.

q = You are a computer science major

r = You are the freshman

$\Rightarrow (q \vee \neg r) \rightarrow p$

Construct the truth table:

$((p \leftrightarrow q) \oplus (\neg p \rightarrow q)) \vee (q \rightarrow \neg r)$ (*)

p	q	r	$\neg p$	$\neg r$	$p \leftrightarrow q$	$\neg p \rightarrow q$	$q \rightarrow \neg r$	(*)
T	T	T	F	F	T	T	F	T
T	T	F	F	T	T	T	F	T
T	F	T	F	F	F	T	T	T
T	F	F	F	T	F	T	T	T
F	T	T	T	F	F	T	T	T
F	T	F	T	T	F	T	F	T
F	F	T	T	F	T	F	T	T
F	F	F	T	T	T	F	T	T

6. Biconditional

Given propositions p and q, the biconditional $p \leftrightarrow q$ is a proposition that is true when p & q have same truth values. Alternatively $p \leftrightarrow q$ is true whenever both $p \rightarrow q$ and $q \rightarrow p$ are true. Some of the technologies used for biconditional are:-

1. "p if and only if q"
2. "if p then q and coversly"
3. "p is necessary and sufficient for q".

Eg:- p = "today is sunday"
q = "it is hot today"

$p \leftrightarrow q$: today is sunday if and only if it is hot day.

p	q	$p \rightarrow q$	$q \rightarrow p$	$p \leftrightarrow q$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

let p, q and r be the proposition with truth value T, F, T respectively. Evaluate the following:

i) $\neg r \vee \neg(p \vee q)$
ii) $\neg(p \vee q) \wedge (\neg r) \vee q$

i) $\neg r \vee \neg(p \vee q)$

p	q	r	$\neg r$	$p \vee q$	$\neg(p \vee q)$	$\neg r \vee \neg(p \vee q)$
T	F	T	F	F	T	F

(ii) $\neg(p \vee q) \wedge (\neg r) \vee q$

p	q	r	$\neg r$	$p \vee q$	$\neg(p \vee q)$	$(\neg r) \vee q$	$\neg(p \vee q) \wedge (\neg r) \vee q$
T	F	T	F	F	T	F	F

Tautology and Contradiction

A compound proposition that is always true, no matter what the truth values of the atomic propositions that contain it is called tautology.

For example:-

$p \vee \neg p$ is always true verify

p	$\neg p$	$p \vee \neg p$
T	F	T
T	F	T
F	T	T
F	T	T

A compound proposition that is always false is called contradiction. For eg:- $p \wedge \neg p$.

p	$\neg p$	$p \wedge \neg p$
T	F	F
F	T	F

A compound proposition that is neither a tautology nor a contradiction is called contingency.

Logical Equivalences:

The compound propositions p and q are logically equivalent denoted by $p \Leftrightarrow q$ or $p \equiv q$, if proposition $p \leftrightarrow q$ is tautology.

Some important logical equivalences:-

1. $p \wedge T \Leftrightarrow p$ — Identity Law
2. $p \vee F \Leftrightarrow p$ — ,,
3. $p \wedge F \Leftrightarrow F$ — Domination Law
4. $p \vee T \Leftrightarrow T$ — ,,
5. $p \wedge p \Leftrightarrow p$. — Idempotent law
6. $p \vee p \Leftrightarrow p$ — ,,
7. $\neg(\neg p) \Leftrightarrow p$ — Double negation law
8. $p \wedge q \Leftrightarrow q \wedge p$ — Commutative law
9. $p \vee q \Leftrightarrow q \vee p$ — ,,
10. $(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$ — Associative law
11. $(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$ — ,,
12. $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$ — Distributive law
13. $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$ — ,,
14. $\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$ — De-Morgan's Law
15. $\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$ — ,,

16. $p \wedge \neg p \Leftrightarrow F$ Trival Tautology

17. $p \vee \neg p \Leftrightarrow T$ "

18. $p \to q \Leftrightarrow \neg p \vee q$ Implication

19. $p \leftrightarrow q \Leftrightarrow (p \to q) \wedge (q \to p)$ Equivalence

20. $(p \wedge q) \to r \Leftrightarrow p \to (q \to r)$ Exportation

21. $(p \to q) \wedge (p \to \neg q) \Leftrightarrow \neg p$ Absurdity

22. $p \to q \Leftrightarrow \neg q \to \neg p$ Contrapositive

23. $p \wedge (p \vee q) \Leftrightarrow p$ Absorption

24. $p \vee (p \wedge q) \Leftrightarrow p$ "

① Truth Table
② Symbolic Derivation

12. $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$

$p \wedge (q \vee r)$

p	q	r	$q \vee r$	$p \wedge (q \vee r)$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	F	F
F	T	T	T	F
F	T	F	T	F
F	F	T	T	F
F	F	F	F	F

$(p\wedge q)\vee(p\wedge r)$

p	q	r	$p\wedge q$	$p\wedge r$	$(p\wedge q)\vee(p\wedge r)$
T	T	T	T	T	T
T	T	F	T	F	T
T	F	T	F	T	T
T	F	F	F	F	F
F	T	T	F	F	F
F	T	F	F	F	F
F	F	T	F	F	F
F	F	F	F	F	F

Hence, $p\wedge(q\vee r) \Leftrightarrow (p\wedge q)\vee(p\wedge r)$

Prove that:- $(p\rightarrow q)\wedge(p\rightarrow \neg q) \equiv \neg p$ by the help of truth table.

p	q	$\neg q$	$(p\rightarrow q)$	$(p\rightarrow \neg q)$	$(p\rightarrow q)\wedge(p\rightarrow \neg q)$
T	T	F	T	F	F
T	F	T	F	T	F
F	T	F	T	T	T
F	F	T	T	T	T

$\neg p$
F
F
T
T

Hence, $(p\wedge q)\wedge(p\rightarrow \neg q) \equiv \neg p$

Show that $\neg(p \to q)$ and $(p \wedge \neg q)$ are logically equivalent by the help of symbolic derivation.

Solution:-

$LHS: \neg(p \to q)$

$\equiv \neg(\neg p \vee q)$ [∵ Implication law]

$\equiv \neg(\neg p) \wedge \neg(q)$ [∵ Demorgan's law]

$\equiv p \wedge \neg q$ [∵ double negation]

$\equiv RHS$

proved!

$(p \to q) \wedge (p \to \neg q) \equiv \neg p$

$LHS \equiv (p \to q) \wedge (p \to \neg q)$

$\equiv (\neg p \vee q) \wedge (\neg p \vee \neg(\neg q))$ [Implication law]

$\equiv [\neg p \vee q] \wedge [\neg p \vee q]$ ~~[double negation]~~

$\equiv \neg p$ [Absurdity law.

Show that $\neg(p \vee (\neg p \wedge q))$ and $(\neg p \wedge \neg q)$ are logically equivalent

$LHS \equiv \neg(p \vee (\neg p \wedge q))$

$\equiv \neg p \wedge \neg(\neg p \wedge q)$ De Morgan's Law

$\equiv \neg p \wedge (p \vee \neg q)$ double negation

$\equiv$ ~~$\neg p \wedge p \vee \neg q$~~

~~$\equiv F \vee \neg q$~~

$\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q)$ (Distributive ~~Double Negation~~)

$\equiv F \vee (\neg p \wedge \neg q)$ [Trit tautology]

$\equiv (\neg p \wedge \neg q) \vee F$ (commutative) $\equiv \neg p \wedge \neg q$ (Identity

LHS:- $\neg(p \vee (\neg p \wedge q))$

$\equiv \neg[(p \vee \neg p) \wedge (p \vee q)]$ (distributive law)

$\equiv \neg(T \wedge (p \vee q))$ Tri. tautology).

$\equiv \neg((T \wedge p) \vee (T \wedge q))$ Distributive

$\equiv \neg((p \wedge T) \vee (q \wedge T))$ Commutative

$\equiv \neg(p \vee q)$ Identity

$\equiv \neg p \wedge \neg q$ De Morgan's

proved!

Show that $(p \wedge q) \to (p \vee q)$ is a tautology.

$\equiv (p \wedge q) \to (p \vee q)$

$\cancel{\equiv (\neg p \vee q) \to}$

$\equiv \neg(p \wedge q) \vee (p \vee q)$ [Implication Law]

$\equiv (\neg p \vee \neg q) \vee (p \vee q)$ [De Morgans Law)

$\equiv (\cancel{\neg p \vee p})$

$(p \wedge q) \to (p \vee q) \equiv \neg(p \wedge q) \vee (p \vee q)$ [Implication Law

$\equiv (\neg p \vee \neg q) \vee (p \vee q)$ [De Morgan law

$\equiv (\neg p \vee p) \vee (\neg q \vee q)$ [Associative.

$\equiv (p \vee \neg p) \vee (q \vee \neg q)$ [Commutative.

$\equiv T \vee T.$ [Trival. Taulogy]

$\equiv T$

Dual of compound proposition

The dual of compound proposition that contains only the logical operators $\wedge$, $\vee$ and $\neg$ is the compound proposition obtained by replacing each $\vee$ by $\wedge$, each $\wedge$ by $\vee$, each T by F and each F by T. The dual of s is denoted by s^*.

Find the dual of each of these compound proposition.

ⓐ $p \vee \neg q$

$\Rightarrow s = p \vee \neg q$

$s^* = p \wedge \neg q$

ⓑ $p \wedge (q \vee (r \wedge T))$

$s = p \wedge (q \vee (r \wedge T))$

$s^* = p \vee (q \wedge (r \vee F))$

ⓒ $(p \wedge \neg q) \vee (q \wedge F)$

$s = (p \wedge \neg q) \vee (q \wedge F)$

$s^* = (p \vee \neg q) \wedge (q \vee T)$

ⓓ $(p \vee F) \wedge (q \vee T)$

$s = (p \vee F) \wedge (q \vee T)$

$s^* = (p \wedge T) \vee (q \wedge F)$

When does $S^* = S$, where S is a compound proposition?

Solution:-

Since S is a compound proposition, Let p be proposition then

$$S = p \vee p$$

The dual of $S \Rightarrow S^* = p \wedge p$

$$p \vee p = p \wedge p = p \text{ (Idempotent law)}$$

$$\therefore S = S^*$$

Show that $(S^*)^* = S$ where S is a compound proposition.

Solution:-

S is a compound proposition then

Let $S = p \wedge \neg q$

Now,

$$S^* = p \vee \neg q$$

$$(S^*)^* = p \wedge \neg q$$

$$\therefore (S^*)^* = S$$

proved!

Predicate.

Lets take a statement $x>3$, there are two parts one is the variable part called "subject" and another is relation part ">3" called predicate. We can ~~den~~ denote the statement "$x>3$" by $P(x)$ where P is a predicate ">3" and x is the variable. Once the value is assigned to the propositional function then we can tell whether it is true or false i.e. proposition.

→ The logic involving predicates is called predicate logic or predicate calculus.

Let $P(x)$ denotes the statements "$x>10$". What are the truth value of $P(12)$ and $P(5)$?

Solution:-

~~$P(x) = x>10$~~

~~$P(12) = 12>10$ (Truth)~~

$P(5) = 5 \not> 10$ (false)

$P(x)$ denotes $x>10$

$P(12)$ denotes $12>10$ which is "true"

$P(5)$ denotes $5>10$ which is "false"

Let Q (x, y) denote the statement "$x = y+3$". What are the truth value for the propositions Q(1, 2) and Q (3, 0)?

Solution:-

Q(x, y) denotes $x = y+3$

Q(1, 2) denotes $1 = 2+3$
which is "False"

Q (3, 0) denotes $3 = 0+3$
which is "True"

Let R(x, y, z) denote the statement "$x+y = z$". What are the truth value of R(1, 2, 3) & R (0, 0, 1)?

Solution:-

R (x, y, z) denotes $x+y = z$

R(1, 2, 3) denotes $1+2 = 3$
which is "True"

R(0, 0, 1) denotes $0+0 = 1$
which is "False"

Quantifiers

Quantifiers are the tools to make the propositional function a proposition. Construction of propositions from the predicates using quantifiers is called quantification. The variables that appear in the statement can take different possible values and all the possible values that the variables can take forms a domain called "Universe of Discourse" or "Universal set".

Types of Quantifiers:

1. Universal Quantifier
2. Existential Quantifier

Universal Quantifier

It is denoted by $(\forall)$ symbol "for all". The universal Quantification of $P(x)$ denoted by for all x $P(x)$ is

$\forall x \, P(x)$

is a proposition. "$P(x)$ is true for all the values of x in the Universe of Discourse".

We can represent the universal quantification by using the english sentences like (i) "for all x $P(x)$ holds".

(ii) "for every x $P(x)$ holds."

(iii) "for each x $P(x)$ holds".

Example:-

Express the statement into quantified statement. "all students of CSIT takes discrete mathematics class", where UoD is set of all CSIT students".

$\Rightarrow$ D(x) denotes x takes discrete mathematics class.

$\forall x \; D(x)$

D(x) = predicate

Existential Quantifiers

It is denoted by "$\exists$". The existential Quantification of p(x) denoted by there exists x P(x) "$\exists x P(x)$".

"P(x) is true for some values of x in UoD".

It can be represented like

i) "there exists x such that P(x) is true".

ii) "P(x) is true for at least one x."

Examples:-

Let Q(x) be the statement "$x < 2$". What is truth value of the quantification $\forall x Q(x)$ where domain consists of all real numbers?

$\Rightarrow$ Q(x) denotes "$x < 2$"

UoD = { ... -2, -1, 0, 1, 2 ... }

$x = 3$

$\forall x Q(x) \Rightarrow Q(3) \Rightarrow 3 > 2$

False ∴ The truth value...

What is the truth value of $\exists x\, P(x)$ where $P(x)$ is the statement "$x^2 > 10$" and the UoD consists of the integer not exceeding 4?

$\Rightarrow$ $P(x)$ denotes "$x^2 > 10$"

UoD = $\{0, 1, 2, 3, 4\}$

$\exists x\, P(x)$ (True)

Because: $P(2)$.

$4^2 > 10$

$16 > 10$

(True)

∴ The true value of $\exists x\, P(x)$ is true.

Free and Bound Variables

When the variable is assigned a value or it is quantified. It is called Bound variable. If the variable is not bounded then it is called free variables

Examples:-

Identify the free and bound variables:

1. $P(x, y)$, both are free variables.
2. $P(2, y)$, $\Rightarrow$ y is free.
3. $P(2, y)$ where $y = 4$ $\Rightarrow$ both bounded.
4. $\forall x\, P(x)$, x is bounded variable
5. $\forall x\, P(x, y)$, x bounded, y free

Expression with no free variable is **proposition**?

Expression with at least one free variables is **predicate**?

Order of Quantification:-

Example:-

Let $L(x, y)$ denotes x loves y where U.O.D for x, y is set of all people in the world. Translate the given quantified statement in English.

i) $\forall x \exists y \, L(x, y)$

→ for all x there exist some y such that x loves y. i.e. Everyone loves someone.

ii) $\exists y \, \forall x \, L(x, y)$

→ There exist some y for all x such that x loves y. i.e. Someone is loved by everyone.

iii) $\forall x \, \forall y \, L(x, y)$

⇒ for all x and y such that x loves y i.e. everyone loves everybody

iv) $\exists x \, \exists y \, L(x, y)$

⇒ there exist some ~~y~~ x and some y such that x loves y

someone loves somebody

Negation of Quantified Expression.

$\forall x\, P(x) \Rightarrow \neg \forall x\, P(x) \equiv \exists x\, \neg P(x)$

$\exists x\, P(x) \Rightarrow \neg \exists x\, P(x) \equiv \forall x\, \neg P(x)$,

Example:-

Let $P(x)$ denotes x is lovely, UoD for x is girls in Kathmandu.

$\forall x\, P(x)$

$\Rightarrow$ Every girl in Kathmandu are lovely

$\exists x\, P(x)$

$\Rightarrow$ Some girls in Kathmandu are lovely

$\neg \forall x\, P(x)$

$\Rightarrow$ Not all girls in Kathmandu are lovely

$\neg \exists x\, P(x)$

$\Rightarrow$ All girls in Kathmandu are not lovely

Translate the sentence into logical expression.

1. "not every integer is even.

let $P(x)$ denotes x is even, UoD for integes integer

$\neg \forall x\, P(x)$

Translate " If a person is female and is a parent then this person is someone's mother". into logical expression where UoD is set of all peoples.

$\Rightarrow$ Let $F(x)$ denotes female
$P(x)$ denotes parent

UoD = { set of all people }
$M(x, y)$ denotes x is mother of y.

$$\forall x\, \exists y\, ((F(x) \land P(x)) \rightarrow M(x, y))$$

classmate
Date
Page

Mathematical Reasoning.

Rules of Reasonining

To draw conclusion from the given premise we must be able to apply some well defined steps that helps ~~reaching~~ reaching the conclusion. These steps of reaching the conclusion are provided by rule of inference

Rule1: Modus Ponens (or Law of Detachment)

Whenever two propositions p & $p \to q$ are both true then we confirm q is true i.e.

$$\frac{\begin{array}{c} p \\ p \to q \end{array}}{q.}$$

, this rule is value rule of inference because the implication $[p \wedge (p \to q)] \to q$ is a tautology.

p	q	$p \to q$	$p \wedge (p \to q)$	$[p \wedge (p \to q)] \to q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Rule2: Hypothetical Syllogism (Transitive Rule)

Whenever two proposition $p \to q$ and $q \to r$ are both true then we confosm that implication $p \to r$ is true.

i.e. $$\frac{\begin{array}{c} p \to q \\ q \to r \end{array}}{\therefore p \to r}$$

, this rule is valid rule of inference because the implication

$[(p \to q) \land (q \to r)] \to (p \to r)$ is tautology.

Similarly,

$$\begin{array}{l} p \to q \\ q \to r \\ r \to s \\ \hline \therefore p \to s \end{array}$$

p	q	r	$p \to q$	$p \to r$	$q \to r$	$[(p \to q) \land (q \to r)]$	$[(p \to q) \land (q \to r)] \to (p \to r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	T	T	T
T	F	F	F	F	T	T	T
F	T	T	T	T	T	T	T
F	T	F	T	T	F	F	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

Rule 3 :- Addition

Due to tautology $p \to (p \lor q)$, rule $\dfrac{p}{\therefore p \lor q}$ is valid rule of inference.

$p \to (p \lor q)$

$\equiv \neg p \lor (p \lor q)$ (implication)

$\equiv \neg p \lor p \lor q$

$\equiv T \lor q$ (Trival tautology)

$\equiv T$ [Domination]

$[(p\to q)\wedge(q\to r)]\to(p\to r)$

$\equiv \neg[(p\to q)\wedge(q\to r)]\vee(p\to r)$ [∵ Implication law]

$\equiv \neg[(p\to q)\wedge(q\to r)]\vee(\neg p\vee r)$ [∵ Implication law]

$\equiv \neg[p\to q]\vee\neg(q\to r)\vee(\neg p\vee r)$ [∵ De Morgan's law].

$\equiv \neg(\neg p\vee q)\vee\neg(\neg q\vee r)\vee(\neg p\vee r)$ [Implication law]

$\equiv (\neg(\neg p)\wedge\neg q)\vee(\neg(\neg q)\wedge\neg r)\vee(\neg p\vee r)]$ [De Morgan's law]

$\equiv (p\wedge\neg q)\vee(q\wedge\neg r)\vee(\neg p\vee r)$ [Double Negation]

$\equiv$

$[(p\to q)\wedge(q\to r)]\to(p\to r)$

$\equiv (p\wedge\neg q)\vee(q\wedge\neg r)\vee(\neg p\vee r)$ [Implication]

$\equiv (p\wedge\neg q)\vee\{(q\wedge\neg r)\vee r\}\vee\neg p$ [Associative)

$\equiv (p\wedge\neg q)\vee\{(q\vee r)\wedge(\neg r\vee r)\}\vee\neg p$ [Distributive)

$\equiv (p\wedge\neg q)\vee\{(q\vee r)\wedge T\}\vee\neg p$ [Trival Tautology]

$\equiv \{(p\wedge\neg q)\vee\neg p\}\vee(q\vee r)\wedge T$ [Associative].

$\equiv (p\vee\neg p)\wedge(\neg q\vee\neg p)\vee(q\vee r)\wedge T$ (Distributive)

$\equiv T\wedge(\neg q\vee\neg p)\vee(q\vee r)\wedge T$ (Trival tautology).

$\equiv T\wedge(\neg q\vee q)\vee\neg p\vee r\wedge T$ (Associative)

$\equiv T\wedge\{T\vee(\neg p\vee r)\}\wedge T$ (Dominative)

$\equiv T\wedge T\wedge T$ (Trival tautology)

$\equiv T$

Rule 4 :- Simplification :-

Due to the tautology $(p \wedge q) \rightarrow p$, rule $\frac{p \wedge q}{\therefore p}$ is a valid rule of inference.

$$(p \wedge q) \rightarrow p$$
$\equiv \neg(p \wedge q) \vee p$ [Implification]
$\equiv \neg p \vee \neg q \vee p$ (De-Morgan's Law]
$\equiv p \vee \neg p \vee \neg q$
$\equiv T \vee \neg q$ (Trival tautology)
$\equiv T$ (Trival tautology)

Rule 5 : Conjuction

Due to the tautology $[(p) \wedge (q)] \rightarrow (p \wedge q)$; rule

p
q is valid rule of inference.
$\therefore p \wedge q$

$$[(p) \wedge (q)] \rightarrow (p \wedge q)$$
$\equiv (p \wedge q) \rightarrow (p \wedge q)$
$= \neg(p \wedge q) \vee (p \wedge q)$ [Implication]
$\equiv \neg p \vee \neg q \vee (p \wedge q)$ [De-Morgan's law]

Rule 6: Modes Tollens

Due to tautology $[\neg q \wedge (p \to q)] \to \neg p$, rule $\dfrac{\begin{array}{l}\neg q \\ p \to q\end{array}}{\therefore \neg p}$

is valid rule of inference.

p	q	$\neg p$	$\neg q$	$(p \to q)$	$\neg q \wedge (p \to q)$	$\neg q \wedge (p \to q) \to \neg p$
T	T	F	F	T	F	T
T	F	F	T	F	F	T
F	T	T	F	T	F	T
F	F	T	T	T	T	T

$\neg q \wedge (p \to q) \to \neg p$

Rule 7: Disjunctive Syllogism

Due to tautology $[(p \vee q) \wedge \neg p)] \to q$, rule $\dfrac{\begin{matrix} p \vee q \\ \neg p \end{matrix}}{\therefore q}$ is a valid rule of inference.

p	q	$\neg p$	$p \vee q$	$(p \vee q) \wedge \neg p$	$((p \vee q) \wedge \neg p) \to q$
T	T	F	T	F	T
T	F	F	T	F	T
F	T	T	T	T	T
F	F	T	F	F	T

$[(p \vee q) \wedge \neg p] \to q$

Rule 8: Resolution

Due to tautology $[(p \vee q) \wedge (\neg p \vee r)] \rightarrow (q \vee r)$, rule

$$\begin{array}{l} p \vee q \\ \neg p \vee r \\ \hline \therefore q \vee r \end{array}$$

is valid rule of inference

$[(p \vee q) \wedge (\neg p \vee r)] \rightarrow (q \vee r)$

Example:-

For the set of premises "If I play football, then I am sore the next day", "I will take rest if I am sore", "I did not take rest". What relevant conclusion can be drawn? Explain the rule of inference to draw the conclusion.

Solution:-

p = I play football

q = I am sore

r = I ~~to~~ will take rest

Hypothesis:-

(i) $p \to q$

(ii) $q \to r$

(iii) $\neg r$

Steps	Reason.
(i) $p \to q$	Hypothesis.
(ii) $q \to r$	Hypothesis
(iii) $p \to r$	Transitive from (i) & (ii)
(iv) $\neg r$	Hypothesis
(v) $\neg p$	Modes Tollen

Conclusion:

Therefore we can conclude that, "I did not play football.

Example:-

Show that the hypothesis "It is not sunny this afternoon and it is ~~colder~~ cooler than yesterday". "We will go swimming only if it is sunny", "If we do not go swimming, then we will take a trip" and "If we take a trip, then we will be home by sunset" Lead to the conclusion "We will be home by sunset".

Solution:-

p = It is sunny this afternoon.
q = It is cooler
r = We will go swimming
s = We will take a trip
t = We will be home by sunset.

Hypothesis

(I) $\neg p \wedge q$
(II) $r \to p$
(III) $\neg r \to s$
(IV) $s \to t$

Steps		Reason
(I)	$\neg p \wedge q$	Hypothesis
(II)	$r \to p$	Hypothesis
(III)	$\neg r \to s$	Hypothesis
(IV)	$s \to t$	Hypothesis
(V)	$\neg r \to t$	Transitive from (III) & (IV)
(VI)	$\neg p$	Simplification (I)
(VII)	$\neg r$	(Modus Tollens) (VI) & (II)
(VIII)	s	Modus Ponen (VII) & (III)
(IX)	t	Modus Ponen (VIII) & (IV)

Fallacies:-

The fallacies are arguement that are convincing but not correct. So fallacies produce faulty inference. Fallacies are contingency rather than tautology.

(1) Fallacy of affirming the conclusion:

The kind of fallacy has the form

$$\begin{array}{l} q \\ p \rightarrow q \\ \hline \therefore p \end{array}$$

i.e. $q \wedge (p \rightarrow q) \rightarrow p$. This is not a tautology, hence it is fallacy.

p	q	$p \rightarrow q$	$q \wedge (p \rightarrow q)$	$q \wedge (p \rightarrow q) \rightarrow p$
T	T	T	T	T
T	F	F	F	T
F	T	T	T	F
F	F	T	F	T

Example:-

"If the economy of Nepal is poor, then the education system in Nepal will be poor" "The education system in Nepal is poor. Therefore, "Economy of Nepal is poor".

Solution:-

p = economy of Nepal is poor

q = education system in Nepal is poor.

$$\begin{array}{l} p \rightarrow q \\ q \\ \hline \therefore p \end{array}$$

(which is fallacy (fallacy of affirming the conclusion)) Hence we can conclude economy of Nepal is not poor.

(iii) Fallacy of denying the hypothesis

This kind of fallacy has the form $\frac{\begin{matrix}\neg p \\ p \to q\end{matrix}}{\therefore \neg q}$

i.e. $[\neg p \wedge (p \to q)] \to \neg q$. This is not a tautology hence it is a fallacy.

p	q	$\neg p$	$\neg q$	$p \to q$	$\neg p \wedge (p \to q)$	$[(\neg p \wedge (p \to q))] \to \neg q$
T	T	F	F	T	F	T
T	F	F	T	F	F	T
F	T	T	F	T	T	F
F	F	T	T	T	T	T

Example:-

"If today is sunday, then it rains today". & "Today is not sunday". Therefore, "It doesnot rain today".

$\Rightarrow$ p : today is sunday

q = it rains today.

$$\frac{\begin{matrix} p \to q \\ \neg p \end{matrix}}{\therefore \neg q}$$

(iii) Begging the question (Circular Reasoning)

If the statement that is used for loop is equivalent to the statement that is being proved then It is called circular reasoning.

Example:-
Ram is black because he is black.

Rules of Inference for Quantified Statement

Universal Instantiation:-
If the proposition of the form $\forall x\, P(x)$ is supposed to be true then the Universal quantifier can be dropped out to get $P(c)$ is true for arbitary c in the universe of discourse.

i.e. $\dfrac{\forall x\, P(x)}{\therefore P(c), \text{ for all } c}$

Universal Generalization:-
If all the instances of c makes $P(c)$ true, then $\forall x\, P(x)$ is true. This can be written as.

$$\frac{P(c) \text{ for all } c}{\therefore \forall x\, P(x)},$$

Here the choosen c must be arbitrarily not a specific element from the UoD.

Date ______
Page ______

Existential Instantiation:

If the proposition of the form $\exists x\, P(x)$ is supposed to be true then there is an element x in the UoD such that P(c) is true.

i.e. $\dfrac{\exists x\, P(x)}{\therefore P(c) \text{ for some } c}$

Here x is not an arbitary, it must be scientific such that p(x) is true.

Extential Generalization

If at least an element c from the UoD make P(c) true then $\exists x P(x)$ is true.

i.e. $\dfrac{P(c), \text{ for some } c}{\therefore \exists x\, P(x)}$

Show that the premises, "Everyone in this discrete mathematics class taken a course in computes Science" and "Marla is a student in this class" imply the conclusion "Marla has take a course in Computes Science".

Solution:-

D(x) denotes x is in discrete mathematics class

C(x) denotes x has taken a course in Cs.

Quantified statement (Hypothesis)

1. $\forall x\, (D(x) \rightarrow C(x))$
2. D(Marla)

③ C (Marla)

Steps	Result :

Steps	Result
① $\forall x(D(x) \rightarrow C(x))$	Hypothesis.
② $D(Marla) \rightarrow C(Marla)$	Universal Instantiation from ①.
③ D(Marla)	Hypothesis
④ C(Marla)	Modes ponens from ② & ③.

Prove or disprove the validity of the arguement, "every living thing is a plant or animal", "Hari's dog is alive and it is not a plant.", "All animals have heart", hence "Hari's dog has a heart".

Solution:- UoD denotes "every living thing is a plant or animal".

H(x) denotes All animals have heart.

$P(x) \rightarrow x$ is plant, $A(x) \rightarrow x$ is animal, $J(x) \rightarrow x$ is alive.

1. $\forall x(P(x) \vee A(x))$ — Hypothesis.
2. J(Haris dog) $\wedge \neg$P(Haris dog) — Hypothesis.
3. J(Haris dog) $\wedge$ A(Haris dog) — from ① & ②.
4. $\forall x\, H(x)$ — Hypothesis
5. H(Haris Dog) — Universal instation from (5)

Hypothesis:
(i) $\forall x(P(x) \vee A(x))$
(ii) J(Haris Dog) $\wedge \neg$p(Haris Dog)
(iii) $\forall x\, H(x)$
(IV) H(Haris dog).

Proving Theorem

1. Direct Proof:-

We prove the implication $p \to q$, where we start assuming that the hypothesis i.e. p is true and using rule of inference, theorems etc. If q becomes true, then the arguement becomes valid. This is known as direct proof.

Example:-

If a and b are odd integer, then a+b is even integer.

Solution:

By the def^n of odd integer
$\rightarrow n = 2k+1$

By the def^n of even integer.
$\rightarrow n = 2k.$

Here,

$a = 2k+1$ ($\because$ By def^n)

$b = 2l+1$ ($\because$ " ")

So,

$a+b = 2k+1+2l+1$

$= 2k+2l+2$

$= 2(k+l+1)$

$= 2m$ ($\because (k+l+1) = m$ is any integer).

$\therefore$ $a+b = 2m$ which is given by the def^n of even integer.

Example :-

If m, n are divisible by 3, mn is divisible by 9.

Solution:-

By the defination of divisible by 3

$\rightarrow m, n / 3 = \{1, 2, 3 \ldots\}$

By the defination of divisible by 9

$\rightarrow m, n / 9 = \{1, 2, 3 \ldots\}$

Here,

$m = 3k$

$n = 3l$

So,

$mn = 3k \cdot 3l$

$= 9(kl)$

$= 9n$ ($\because kl = n$ is any integer).

Show that the square of an even number is an even number using direct proof.

Solution:-

By defination of ~~sque~~ even number.

$n = 2k$

By defination of square of even number.

$n^2 = (2k)^2$

Here,

$n = 2k$ from defination.

So.

$n = 2k$

Squaring we get.

$n^2 = 4k^2$ (where k is an even integer).

$= 4m$

$\therefore n^2 = 4m$ ($\because k^2 = m$, which is given by def^n of square of even num)

Use direct proof to show that every odd integer is the difference of two square.

Sol^n:-

By defination of odd integer
$n = 2k+1$.

By defination of ~~difference~~ square.
$n^2 = (2k+1)^2$

∴ Now,

$$\cancel{n = (2k+1)^2 - (2l+1)^2}$$
$$\cancel{= 4k^2 + 4k + 1 - 4l^2 - 4l - 1}$$
$$\cancel{= 4k^2 + 4k - 4l^2 - 4l}$$

Indirect Proof (Proof by Contraposition)

If we prove the implication $p \to q$ by assuming the conclusion is false i.e. $p \to q \equiv \neg q \to \neg p$ is known as proof by contraposition or indirect proof.

Example:-

If the product of two integer a & b is even, then either a is even or b is even

Solution:-

$\neg q$ = a and b are both odd.

$\neg p$ = product of two integer a & b is odd.

we know,

$a = 2k+1$

$b = 2k+1$

Now,
$ab = (2k+1)(2l+1)$
$= \cancel{2k+2l+2} \; 4kl + 2k + 2l + 1$
$= \cancel{2(k+l+1)} \; 2(2kl + k + l) + 1$
$= 2m+1$ ($m = 2kl + k + l$; m is any integer)

we know that.
$\neg q \to \neg p \equiv p \to q$

$\therefore$ We can conclude that product of two integer a & b is even, then either a is even or b is even.

Prove that if n is an integer and 3n+2 is odd, then n is odd.

Solution:-
~~By definition of integer.~~
$\neg p = n$ is ~~not~~ ~~odd~~. even
~~By~~. $\neg q = (3n+2)$ is even.

We know,
$n = \cancel{2k+1} \; 2k$

Now,
$\cancel{3n+2} = 3(\cancel{2k+1}) + 2$ $\quad \cancel{3n+2} = 3(\cancel{k}) + 2$
$= \cancel{6k+3+2}$ $\quad = \cancel{3k+2}$
$= \cancel{6k+5}$ $\quad = \cancel{3k+1+1} \; \cancel{2k+k+1+1}$
$\quad = \cancel{(2k+1)+(k+1)}$

$3n+2 = 3(2k) + 2$
$= 6k + 2$
$= \cancel{3k+5} \; 2(k+2) = 2m$

classmate
Date
Page

Proofs by Contradiction

The steps in proof of implication $p \to q$ by contradiction are:-

i) Assume $p \wedge \neg q$ are true.

ii) Try to show the above assumption is false

Example:-

If a^2 is an even number, then a is an even number.

Solution:-

Assume a is an odd number and a^2 is an even number.

$a = 2K+1$ (By def^n of odd integer)

then,

$a^2 = 4K^2 + 4K + 1$

$= 2(2K^2+2K)+1$

$= 2m+1$ (where $2K^2+2K = m$; is any integer)

So, by contradiction, we can prove that If a is an even number then a^2 is an even number.

Give a proof by contradiction of the theorem "If $3n+2$ is odd, then n is odd".

Sol^n:-

Assume that n is ~~odd~~ even number and $3n+2$ is odd.

By def^n of even integer.

$n = 2K$

Then,

$3n+2 = 3(2K)+2$

$= 6K+2$

$= 2(3K+1)$

$= 2m$ $(3K+1 = m;$ is any integer$)$

So, by contradiction, we can prove that, if $3n+2$ is odd, then n is odd.

Show that if n is an integer and n^3+5 is odd, then n is even using

i) a proof by contraposition

ii) a proof by contradiction

Soln:

(i) by contraposition

$n \Rightarrow$ integer,

$\neg p \Rightarrow n^3+5$ is even.

$\neg q \Rightarrow n$ is odd.

Then,

$n = 2K+1$ (By def^n of even integer)

$\neg q = 2K+1,$

$\neg p = n^3+5$

$= (2K+1)^3+5$

$= 8K^3+6K^2+3K+1+5$

$= 8K^3+12K^2+6K+6$

$= 2(4K^3+6K^2+3K+3)$

$= 2m$ $(4K^3+6K^2+3K+3 = m$; m is an integer

$\therefore$ we know that

$\neg q \rightarrow \neg p \equiv p \rightarrow q$

$\therefore$ We can conclude that if n is an integer and n^3+5 is odd, then ~~n~~ is even.

(ii) by contradiction.

Assume n is odd and n^3+5 is odd.

$n = 2k+1$ (By def^n of odd integer)

then,

$$n^3+5 = (2k+1)^3+5$$
$$= 8k^3 + 12k^2 + 6k + 1 + 5$$
$$= 8k^2 + 12k^2 + 6k + 6$$
$$= 2(4k^3 + 6k^2 + 3k + 3)$$
$$= 2m \quad (m = 4k^3 + 6k^2 + 3k + 3 = m \text{ is any integer}).$$

So by contradiction, we can prove that, if n^3+5 is odd then n is even.

Prove that if n is a ~~even~~ perfect square, then $n+2$ is not a perfect square using.

i) Direct proof method
ii) Indirect proof method
iii) Proof by contradiction

Solution:-

i) Direct proof method.

By def^n of perfect square.

$n = a^2$

Then,

~~$n+2 = (a^2)^2+2$~~

~~$= a^2+2$~~ $\quad n = K^2$

$\therefore \; n+2 = (K^2)+2$

$= K^2+2$

which is not a perfect square.

$\therefore \; n+2 = K^2+2$ is not a perfect square by the defination of perfect square.

ii) Indirect proof method (Contraposition).

$\neg p$ = n is not a perfect square.

$\neg q$ = perfect square.

Then,

$n+2 = (K+2)^2$ (By def^n of perfect square).

$\therefore \; \neg q = (K+2)^2$

$= K^2+4$

$= m^2 \qquad ((K+2)^2 = m^2$; m is any integer)

$\therefore$ we can conclude that if n is a perfect square, then n+2 is not a perfect square.

iii) Proof by contradiction

Assume n^2 is a perfect square and $n+2$ is not a perfect square.

Proof by Cases:-

The implication of the form $(p_1 \vee p_2 \vee \ldots p_n) \to q$ can be proved by using the tautology $(p_1 \vee p_2 \vee \ldots \vee p_n) \to q \longleftrightarrow [(p_1 \to q) \wedge (p_2 \to q) \wedge \ldots \wedge (p_n \to q)]$ i.e. we can show every implication $(p_i \to q)$ true for $i = 1, 2 \ldots n$

Examples:-

if $|x| > 3$, then $x^2 > 9$, where x' is a real number.

Solution :-

Consider two cases.

(i) $-x > 3$

(ii) $x > 3$

Since $|x|$ is an absolute value of x (i), the value of x is x when $x \geq 0$. (ii) the value of x is $-x$ when $x < 0$.

i) if $x > 3$; $x^2 > 9$ which is true.
ii) if $-x < 3$, $x^2 > 9$ which is true.

Hence, $|x| > 3$ then $x^2 > 9$.

Proof of Equivalence
We can prove the equivalence i.e. $p \leftrightarrow q$ by showing $p \rightarrow q$ and $q \rightarrow p$ both.

Example:-
Prove that if n is a positive integer, then n is even iff $7n+4$ is even.

Solution:-
Consider $p = n$ is even
$q = 7n + 4$ is even.

We have,
$n = 2k$ (By def^n of even integer)
then, $q = 7(2k) + 4$
$= 14k + 4$
$= 2(7k + 2)$
$= 2m$ $(7k+2 = m$; m is any integer)

$\therefore p \rightarrow q$ is true.

Again,
$\neg p = n$ is odd.
$q = 7n + 4$ is even.

We know,
$n = 2k + 1$

$\therefore q = 7(2k+1) + 4$
$= 14k + 7 + 4$

$q = 7n + 4$ is even
$p = n$ is even

We have,
$n = 2k$

$q = 7(2k) + 4$
$= 14k + 4$
$= 2(7k + 2)$
$= 2m \quad (7k + 2 = m$; m is any integer$)$

$\therefore q \rightarrow p$ is true.

$\therefore$ We have,
$p \rightarrow q$ is true & $q \rightarrow p$ is also true

$\therefore p \leftrightarrow q$ is true

Existence Proof :-

A proof of a proposition of the form $\exists x\, P(x)$ is called an existence proof.

constructive existence proof:

=> Here some element 'a' is found to show P(a) is true.

non-constructive existence proof:-

=> Here, we do not provide 'a' such that P(a) is true but prove that $\exists x\, P(x)$ is true in different way.

Example:-

Prove by using constructive existence proof that there are 100 consecutive positive integers that are not perfect square.

Solution:

$50^2 = 2500$

$51^2 = 2601$

$\therefore$ The 100 consecutive positive integers that are not perfect square are:-

2501 to 2600.

Proof using non-constructive existence proof that there exists irrational numbers x & y such that x^y is rational.

Sol^n:-

$$\text{let } x = \sqrt{2}, \; y = \sqrt{2}$$

then

$$x^y = (\sqrt{2})^{\sqrt{2}}$$

(i) if $(\sqrt{2})^{\sqrt{2}}$ is rational, we are done.

(ii) if $(\sqrt{2})^{\sqrt{2}}$ is ~~ra~~ irrational.

then

$$x = (\sqrt{2})^{\sqrt{2}}, \; y = \sqrt{2}$$

$$x^y = \left((\sqrt{2})^{\sqrt{2}}\right)^{\sqrt{2}} = 2 \quad \text{which is rational}$$

Vacous Proof:-

A proof of $p \to q$ that uses the fact that p is false.

Example:-

Show that proposition $P(0)$ is true, where $P(n)$ is "If $n > 1$ then $n^2 > n$".

Solution.

Given, $p = n > 1$

$q = n^2 > n$

when $n=0$,

$p(0) = 0>1$ (false)

~~$q(0) = 0^2>0$ (false)~~

$\therefore p = n<1$.

and,

$\therefore q = n^2>n$.

$\therefore$ We canconclude that if $n>1$ then $n^2>n$.

Trivial Proof:-

A proof of $p \to q$ that uses the fact that q is true.

Example:-

Let $P(n)$ be "If a and b are +ve integers with $a \geq b$ then $a^n \geq b^n$", where, the domain consists of all integers. show that $P(0)$ is true.

Solution:-

$p = a \geq b$ where a & b are +ve integers

$q = a^n \geq b^n$

then,

~~when $P(0) = a \geq b$~~

when $n=0$

$P(0) = a^0 \geq b^0 = 1 \geq 1$ (True),

$\therefore$ We can conclude that if a & b are +ve integers with $a \geq b$ then $a^n \geq b^n$.

Mathematical Induction.

Principle of mathematical Induction

Steps:-
(I) Basis Step:- Show $P(n_0)$ is true.
(II) Inductive Hypothesis: Assume $P(K)$ is true for $K=n$
(III) Inductive Step:- Show that $P(K+1)$ is true on the basis of Inductive Hypothesis.

Example:-
Use mathematical induction to prove that
$1+2+\ldots\ldots+n = \frac{n(n+1)}{2}$

Solution
1. Basis steps: $n=0$
$\frac{0(0+1)}{2} = 0$

$n=1$
$\frac{1(1+1)}{2} = 1$ which is true for $n=1$.

2. Inductive hypothesis:
Assume $P(K)$ is true
i.e. $P(K) = \frac{K(K+1)}{2} = \frac{K^2+K}{2}$

$= \frac{K(K+1)}{2} = 1+2\ldots K.$

(3) Inductive step:
$P(K+1) = 1+2+3+\ldots + K+(K+1)$
$= \frac{K(K+1)+(K+1)}{2} = \frac{(K+1)(K+2)}{2}$ (By inductive method.

which is true, hence proved.

Example:-

Use mathematical induction to prove that.

$1+2+2^2+ \ldots 2^n = 2^{n+1}-2$.

Solution:

1. Basis step: $n=1$

$1 = 2^{1+1}-1 = 2^2 - 1 = 3$

$n=0$

$2^{0+1}-1 = 2-1 = 1$

which is true for $n=1$.

2. Inductive hypothesis:-

Assume $P(k)$ is true.

i.e. $P(k) = 2^{k+1}-1$; $k = 1+2+\ldots 2^k$

3. Inductive step.

$P(k+1) = 1+2+\ldots 2^k + 2^{k+1}$

$= 2^{k+1}-1+2^{k+1}$

$= \cancel{2^{k+1}(1-1)}\ 2^k \cdot 2^1 - 1 + 2^k \cdot 2^1$

$= \cancel{4^k - 1 + 4^k}$

$= 4^{k+1}-1$

$= 2^{2(k+1)}-1$

$= 2^{2k+2}-1$

Prove that $1.1! + 2.2! + \dots + n.n! = (n+1)! - 1$, whenever n is an +ve integer.

Solution

1. Basic step:-

$$n = 0$$
$$(0+1)! - 1 = 0$$

$$n = 1$$
$$(1+1)! - 1 = 1$$

which is true for $n = 1$.

2. Inductive hypotheses

Assume $P(K)$ is true

i.e. $P(K) = (1.1! + \dots K.K!)$

$$= (K+1)! - 1$$

3. Inductive steps.

$$P(K+1) = 1.1! + 2.2! + \dots (+K.K!) + (K+1)(K+1)!$$

$$= (K+1)! - 1 + (K+1)(K+1)!$$
$$= (K+1)! \, (1 + (K+1)) - 1$$
$$= (K+1)! \, (K+2) - 1$$
$$= (K+2)! - 1$$ By induction method

Strong Induction (Second principle of Mathematical Induction)

Steps:-

1. Basis steps: Show $P(n_0)$ is true.
2. Inductive hypothesis (strong): Assume $P(K)$ is true for all $n_0 \le K \le n$.
3. Inductive step: Show based on assumption that $P(K+1)$ is true.

Eg:- Prove that 2 divides n^2+n whenever n is a positive integer.

Solution:

1. Basis step:- $n=0$ $P(n_0)=0$

$n=1$

$$p(n_1) = \frac{1^2+1}{2} = \frac{2}{2} = 1$$

which is true for $n=1$.

2. Inductive hypothesis (strong).

Assume $P(K)$ is true for all $n_0 \le K \le n$.

$$\text{i.e. } P(K) = \frac{K^2+K}{2}$$

3. Inductive step:-

$$P(K+1) = \frac{(K+1)^2+(K+1)}{2}$$

$$= \frac{K^2+2K+1+K+1}{2} = \frac{K^2+3K+2}{2}$$

[∵ from inductive hypothesis: K^2+K is divisible by 2]

$$= \frac{K^2+K}{2} + \frac{2(K+1)}{2} = \frac{K^2+K}{2} + 2(K+1)$$

$$= 2K + 2 + 2m \quad (m = K^2+K)$$

$$= 2(m+K+1) = 2l$$

which is divisible by 2.

Hence, it is true, proved!

Recursive Defination:-

Steps:-

1. Basis steps:-

specify the value of the function at base (generally 0 or 1)

2. Recursive step:-

specify the rule for finding the value of function using the value of a function already found.

Example:-

Give a ~~reaso~~ recursive def^n of a sequence $\{a_n\}$; $n=1,2 \ldots n$ if $a_n = 10^n$.

Basis step:-

$n=1$

$a_n = 10^n \therefore a_1 = 10$

Recursive step:-

$a_n = 10\, a_{n-1}$.

$\therefore$ The recursive defination of sequence is $a_n = 10a_{n-1}$

Give a recursive def^n of the sequence $\{a_n\}$, $n=1,2,3\ldots$ if

(a) $a_n = 6n$
(b) $a_n = 2n+1$
(c) $a_n = 5$
(d) $a_n = n^2$

Sol^n:-

(a) $a_n = 6n$

Basis step:-
$n=1$.
$a_n = 6n$; $a_1 = 6$

Recurvice ~~stp~~ steps:-
$a_n = 6+(a_n-1)$

$\therefore$ The recursive defination of sequence is $a_n = 6+(a_n-1)$

(b) $a_n = 2n+1$

Basis step:-
$n=1$
$a_n = 2n+1 \Rightarrow a_1 = 3$

Recursive steps:-
$a_n = 2+(a_n-1)$

$\therefore$ The recursive defination of sequence is
$a_n = 2+(a_n-1)$,

(c) $a_n = 5$

Basis steps:

$n = 1$

$a_n = 5$; $a_1 = 5$.

Recursive steps:-

$a_n = 5$

∴ The ~~term~~ recursive defination of sequence is $a_n = 5$

(d) $a_n = n^2$

Basis steps:-

$n = 1$

$a_n = n^2$; $a_1 = 1$; $a_2 = 4$; $a_3 = 9$ ...

Recursive step:-

$a_n = (2n - 1) + (a_{n-1})$

∴ The recursive defination of sequence is

$a_n = (2n - 1) + (a_{n-1})$

Unit 3:-

Recurrence relation

→ A recurrence relation for a sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence namely $a_0, a_1, \ldots a_{n-1}$ for all integer n with $n \geq n_0$, where n_0 is a non-negative integer.

→ A sequence is called a solution of recurrence relation if its terms satisfies the recurrence relation.

Example:-

For $a_n = 3a_{n-1}$ and $a_0 = 1$ find a_1, a_2, a_3, a_4 & a_5.

Solution:-

$a_0 = 1$

$a_n = 3a_{n-1}$

then

$a_1 = 3 \cdot a_0 = 3$

$a_2 = 3 \cdot a_1 = 9$

$a_3 = 3 \cdot a_2 = 27$

$a_4 = 3 \cdot a_3 = 81$

$a_5 = 3a_4 = 243$

Find the recurrence relation to find the total amount after 30 yrs of a person deposits Rs. 10,000 in a saving account at a bank yielding 11% per year with interest compounded annually.

Soln:-

Basis step:

$n = 1$.

$a_1 = 10,000 \times 11\% + 10,000$

$= 10,000 + 1100$

$= \cancel{1100}\ \cancel{10,}\ 11,100$

$= a_0 + 0.11 a_0$

Let $a_0 = 10,000$

Recursive step:

$\cancel{a_n = (a_{n-1}) - a_0 + 0.11 a_0 \cdot n}$

$a_n = a_{n-1} + 0.11 \times a_{n-1}$

or, $a_n = (1.11)^n P_0$.

$a_1 = 11,100$

$a_2 = 11\%$ of $a_1 + a_1$

$= (1.11)^2 a_0$

$a_3 = (1.11)^3 a_0$

classmate
Date
Page

Solving Recurrence Relations.

1. Linear Homogenous Recurrence Relation of Degree k with constant Coefficients.

Linear Homogenous Recurrence Relation of Degree k with constant coefficient with recurrence relation of the form:-

$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots c_k a_{n-k}$ where, $c_1, c_2 \dots c_k$ are real numbers and $c_k \neq 0$.

The above relation is linear since, right hand side is the sum of the multiples of previous terms of the sequence. It is homogenous because no terms occurs without being multiple of some a_j. All the coefficients of the terms are constant because they do not depend on n. And the degree of the relation is k because a_n is expressed in terms of previous k terms of the sequence.

(i) $P_n = (1.11) P_{n-1}$
(ii) $a_n = a_{n-5}$
(iii) $a_n = a_{n+1} + a^2{}_{n-2}$
(iv) $H_n = 2H_{n-1} + 1$ ⊗ (because its not homogenous)
(v) $B_n = nB_{n-1}$ ⊗ (because there is no constant.

Solving linear Homogeneous Recurrence Relation of Degree K with constant coefficients.

In solving the recurrence relation of this type, the approach is to look for the solution of the form $a_n = r^n$, where r is a constant. $a_n = r^n$ is a solution of a recurrence relation $a_n = C_1 a_{n-1} + C_2 a_{n-2} + \dots + C_n a_{n-k}$. If and only if $r^n = C_1 r^{n-1} + C_2 r^{n-2} + \dots + C_k r^{n-k}$. When we divide both sides by r^{n-k} and transpose the right hand side we have,

$$r^k - C_1 r^{k-1} - C_2 r^{k-2} \dots C_k = 0.$$

Here, we can say $a_n = r^n$ is a solution iff r is the solution of eq^n $r^k - C_1 r^{k-1} \dots C_k = 0$. which is called characteristics of eq^n and the sol^n to this eq^n is called characteristics root.

Theorem 1:-

Let C_1 and C_2 be real numbers. Suppose $r^2 - C_1 r - C_2 = 0$ has two distinct roots r_1 and r_2. Thus the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = C_1 a_{n-1} + C_2 a_{n-2}$ iff $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for $n = 0, 1, 2 \dots$ where α_1 and α_2 are constants.

Example:-

Solve the recurrence relation $a_n = a_{n+1} + 6a_{n-2}$ for $n \geq 2$, $a_0 = 3$, $a_1 = 6$.

Solution:-

Characteristics eq^n:-

$r^2 - C_1 r - C_2 = 0$

$\Rightarrow r^2 - r - 6 = 0$

or, $r_1 = 3$; $r_2 = -2$

The solution of this recourrence rel^n is:

$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$

$\therefore a_n = \alpha_1 3^n + r_2 (-2)^n$

We know that,

$a_0 = 3$

$\alpha_1 (3)^0 + \alpha_2 (-2)^0 = 3$ — (i)

$a_1 = 6$

$\alpha_1 (3)^1 + \alpha_2 (-2)^1 = 6$ — (ii)

from (i) & (ii)

$\alpha_1 + \alpha_2 = 3$

$3\alpha_1 - 2\alpha_2 = 6$

$\therefore \alpha_1 = 12/5$

$\alpha_2 = 3/5$

$\{a_n\}$ is $a_n = \frac{12}{5}(3)^n + \left(\frac{3}{5}\right)(-2)^n$

Exercise

What is the solution of the recurrence relation $a_n = a_{n+1} + 2a_{n-2}$ with $a_0 = 2$ and $a_1 = 7$?

Solution:-

Characteristic eq^n:-

$r^2 - c_1 r - c_2 = 0$

$r^2 - r - 2 = 0$

or, $r^2 - 2r + r - 2 = 0$

or, $r(r-2) + 1(r-2) = 0$

or, $(r-2)(r+1) = 0$

$\therefore r_1 = 2,$

$r_2 = -1$

The solution of the recurrence eq^n is.

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$$

$$\therefore a_n = \alpha_1 (2)^n + \alpha_2 (-1)^n \quad \text{---} (*)$$

We know that,

$$a_0 = 2$$

$$\alpha_1 (2)^0 + \alpha_2 (-1)^0 = 2 \quad \text{---} (i)$$

$$a_1 = 7$$

$$\alpha_1 (2)^1 + \alpha_2 (-1)^1 = 7 \quad \text{---} (ii)$$

From (i) & (ii),

$$\alpha_1 + \alpha_2 = 2 \qquad \text{or, } \alpha_1 = 2 - \alpha_2$$

$$2\alpha_1 - \alpha_2 = 7$$

or, $2(2-\alpha_2) - \alpha_2 = 7$

or, $4 - 2\alpha_2 - \alpha_2 = 7$

or, $4 - 3\alpha_2 = 7$

or, $3\alpha_2 = -3$

or, $\alpha_2 = -1$.

$$\therefore \alpha_1 = 2 - (-1) = 3$$

Putting value of α_1 and α_2, we get.

$$a_n = 3(2)^n + (-1)(-1)^n$$

When $n = 0$

$$a_0 = 3(2)^0 + (-1)(-1)^0 = 3 - 1 = 2$$

$$a_1 = 3(2)^1 + (-1)(-1)^1 = 6 + 1 = 7.$$

∴ The solution of recourrence relation $\{a_n\}$ is $a_n = 3(2)^n + (-1)(-1)^n$.

Exercise:-

Find the explicit formula for the fibonacci numbers. [Use $f_n = f_{n-1} + f_{n-2}$ as recursive defn and $f_0 = 0$ and $f_1 = 1$ as initial condition].

Solution:-

Recursive Relation:-

$$f_n = f_{n-1} + f_{n-2}$$

Characteristic eqn:-

$$r^2 - r - 1 = 0$$

$$\text{or, } r = \frac{\cancel{1} -1 \pm \sqrt{1 - 4 \times (-1)}}{2}$$

$$= \frac{-1 \pm \sqrt{5}}{2}$$

$$\therefore r_1 = \frac{-1+\sqrt{5}}{2} \qquad r_2 = \frac{-1-\sqrt{5}}{2}$$

The soln of recursive relation is.

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$$

$$a_n = \alpha_1 \left(\frac{-1+\sqrt{5}}{2}\right)^n + \alpha_2 \left(\frac{-1-\sqrt{5}}{2}\right)^n$$

We know,

$a_0 = 0$. 1.

$\therefore \alpha_1 = -\alpha_2$ —(1)

$a_1 = 1$

$$\alpha_1\left(\frac{-1+\sqrt{5}}{2}\right) + \alpha_2\left(\frac{-1-\sqrt{5}}{2}\right) = 1$$

$$\text{or, } \frac{-\alpha_1}{2} + \frac{\alpha_1\sqrt{5}}{2} - \frac{\alpha_2}{2} - \frac{\alpha_2\sqrt{5}}{2} = 1$$

$$\text{or, } \cancel{\frac{\alpha_2}{2}} - \frac{\alpha_2\sqrt{5}}{2} - \cancel{\frac{\alpha_2}{2}} - \frac{\alpha_2\sqrt{5}}{2} = 1$$

$$\text{or, } -\alpha_2 = \frac{2}{\sqrt{5}\cdot 2}$$

$$\text{or, } \alpha_2 = \frac{-1}{\sqrt{5}}$$

$$\therefore \alpha_1 = \frac{1}{\sqrt{5}}.$$

Putting value of α_1 & α_2 we get.

$$\{a_n\} \text{ is } a_n = \frac{1}{\sqrt{5}}\left(\frac{-1+\sqrt{5}}{2}\right)^n + \left(\frac{-1}{\sqrt{5}}\right)\left(\frac{-1-\sqrt{5}}{2}\right)^n$$

Theorem 2:

Let C_1 and C_2 be real numbers with $C_2 \neq 0$. Suppose that $r^2 - C_1 r - C_2 = 0$ has only one root r_0. Then the sequence $\{a_n\}$ is a solⁿ of the recurrence relation $a_n = C_1 a_{n-1} + C_2 a_{n-2}$ iff $a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$ for $n = 0, 1, 2 \ldots$ where α_1 and α_2 are constant.

Example:-

Solve the recurrence relation $a_n = 2a_{n-1} - a_{n-2}$ for $n \geq 2$, $a_0 = 3$, $a_1 = 6$

Solution:-

Characteristic eqⁿ:-

$r^2 - 2r + 1 = 0$

or, $r^2 - r - r + 1 = 0$

or, $r(r-1) - 1(r-1) = 0$

or, $(r-1)(r-1) = 0$

$\therefore r_1 = 1;\ r_2 = 1$

The solⁿ of recurrence relation is

$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$

we have, $a_0 = 3$

$\therefore \alpha_1 (1)^0 + \alpha_2 (1)^0 = 3$

or, $\alpha_1 = 3$ — (i).

$a_1 = 6$

$\alpha_1 (1)^1 + \alpha_2 \cdot 1 (1)^1 = 6$

$\alpha_1 + \alpha_2 = 6$

$\alpha_2 = 3$

$\therefore \{a_n\}$ is $a_n = 3(1)^n + 3 \cdot n(1)^n$

What is the solution of the recurrence relation $a_n = 6a_{n-1} - 9a_{n-2}$ with initial condition $a_0 = 1$ & $a_1 = 6$?

Soln:-

Characteristic eqn is

$r^2 - c_1 r - c_2 = 0$

or, $r^2 - 6r + 9 = 0$

or, $r^2 - 3r - 3r + 9 = 0$

or, $r(r-3) - 3(r-3) = 0$

$\therefore (r-3)(r-3) = 0$

$\therefore r_1 = 3$

$r_2 = 3$

The soln of recurrence relation is

$a_n = \alpha_1 (r_1)^n + \alpha_2 n (r_2)^n$

When,

$a_0 = 1$

$\alpha_1 (3)^0 + \alpha_2 0 (3)^0 = 1$

$\alpha_1 = 1$

$a_1 = 6$

$\alpha_1 (3)^1 + \alpha_2 \cdot 1 \cdot (3)^1 = 6$

$3\alpha_1 + 3\alpha_2 = 6$

$\alpha_2 = 1$

$\therefore \{a_n\}$ is $a_n = 1(3)^n + 1n(3)^n$.

Theorem 3:-

Let $c_1, c_2 \ldots c_k$ be real numbers suppose $r^k - c_1 r^{k-1} \ldots c_k = 0$ has k distinct roots $r_1, r_2 \ldots r_k$. Then the sequence $\{a_n\}$ is a solution of the recurrence relation

$$c_n = c_1 a_{n-1} + c_2 a_{n-2} + c_k a_{n-k} \text{ iff}$$

$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \ldots \alpha_k r_k^n$ for $n = 0, 1, 2 \ldots$

where $\alpha_1 \ldots \alpha_k$ are constant.

Example:-

Solve the recurrence relation

$a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$ for $n \geq 3$, $a_0 = 3$, $a_1 = 6$ & $a_2 = 9$

Characteristic Eq^n:-

$$r^3 - 2r^2 - r + 2 = 0$$

or, $r^2(r-2) - 1(r-2) = 0$

or, $(r^2-1)(r-2) = 0$

or, $(r+1)(r-1)(r-2) = 0$

$\therefore r_1 = -1$

$r_2 = 1$

$r_3 = 2$

The sol^n of recurrence relation is:-

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \alpha_3 r_3^n$$

or, $a_n = \alpha_1 (-1)^n + \alpha_2 1^n + \alpha_3 (2)^n$

where,

$a_0 = 3$.

$\alpha_1(-1)^0 + \alpha_2(1)^0 + \alpha_3(2)^0 = 3$

$\alpha_1 + \alpha_2 + \alpha_3 = 3$ — (i)

when $a_1 = 6$.

$\alpha_1(-1)^1 + \alpha_2(1)^1 + \alpha_3(2)^1 = 6$

$-\alpha_1 + \alpha_2 + 2\alpha_3 = 6$ — (ii)

When $a_2 = 9$

$$\alpha_1(-1)^2 + \alpha_2(1)^2 + \alpha_3(2)^2 = 9$$

$$\alpha_1 + \alpha_2 + 4\alpha_3 = 9 \quad \text{--- (iii)}$$

From (i) (ii) & (iii)

$$\alpha_1 + \alpha_2 + \alpha_3 = 3 \quad \text{or, } \alpha_1 + \alpha_2 = 3 - \alpha_3 \quad \text{--- (x)}$$

$$-\alpha_1 + \alpha_2 + 2\alpha_3 = 6$$

$$\alpha_1 + \alpha_2 + 4\alpha_3 = 9$$

from (x)

$$(3 - \alpha_3) + 4(\alpha_3) = 9$$

$$3 - \alpha_3 + 4\alpha_3 = 9$$

$$3 + 3\alpha_3 = 9$$

$$-3 + 9 = 3\alpha_3$$

$$\alpha_3 = 2$$

$\therefore$ from (i) & (ii)

$$\cancel{\alpha_1} + \alpha_2 = 1$$

$$\cancel{-\alpha_1} + \alpha_2 = 2$$

$$2\alpha_2 = 3$$

$$\therefore \alpha_2 = \frac{3}{2}$$

$$\therefore \alpha_1 = 3 - 2 - \frac{3}{2}$$

$$= 1 - \frac{3}{2} = \frac{-1}{2}$$

$\therefore \{a_n\}$ is $a_n = \frac{-1}{2}(-1)^n + \frac{3}{2}(1)^n + 2(2)^n$,

Theorem 4:-

Let $c_1, c_2, \ldots c_k$ be real numbers. Suppose that $r^k - c_1 r^{k-1} \ldots c_k = 0$ has t distinct roots $r_1, r_2 \ldots r_t$ with multiplicity $m_1, m_2 \ldots m_t$ respectively, so that $m_i \geq 1$ for $i = 1, 2, \ldots t$ and $m_1, m_2 + \ldots m_t = k$. Then the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots c_k a_{n-k}$

iff

$$a_n = (\alpha_{1,0} + \alpha_{1,1} n + \ldots + \alpha_{1, m_1 - 1} n^{m_1 - 1}) r_1^n$$

$$+ (\alpha_{2,0} + \alpha_{2,1} n + \ldots + \alpha_{2, m_2 - 1} n^{m_2 - 1}) r_2^n.$$

$$+ \ldots + (\alpha_{t,0} + t_{t,1} n + \ldots + \alpha_{t, m_t - 1} n^{m_t - 1}) r_t^n$$

for $n = 0, 1, 2 \ldots$ where α_{ij} are constant for $1 \leq i \leq t$ and $0 \leq j \leq m_i - 1$.

Example:

Solve the recurrence relation $a_n = 5a_{n-1} - 7a_{n-2} + 3a_{n-3}$ for $n \geq 3$, $a_0 = 1$, $a_1 = 9$ and $a_2 = 15$.

Soln:-

Characteristics eqn:-

$r^3 - 5r^2 + 7r - 3 = 0$

or, $r^3 - r^2 - 4r^2 + 4r + 3r - 3 = 0$

or, $r^2(r-1) - 4r(r-1) + 3(r-1) = 0$

or, $(r-1)(r^2 - 4r + 3) = 0$

or, $(r-1)(r^2 - r - 3r + 3) = 0$

or, $(r-1)\{r(r-1) - 3(r-1)\} = 0$

or, $(r-1)(r-1)(r-3) = 0$

$\therefore r_1 = 1 \quad r_2 = 1 \quad r_3 = 3$

$\therefore r_1 = 1$ and $m_1 = 2$

$r_2 = 3$ and $m_2 = 1$

$$\therefore a_n = (\alpha_{1,0} + \alpha_{1,1} n + \dots + \alpha_{1,m_1-1} n^{m_1-1}) r_1^n + (\alpha_{2,0} + \alpha_{2,1} n + \dots + \alpha_{2,m_2-1} n^{m_2-1}) r_2^n$$

$$a_n = [[(\alpha_{1,0}) + (\alpha_{1,1}) n] 1^n + (\alpha_{2,0}) 3^n]$$

We have

$a_0 = 1$ — (i)

$a_1 = 9$ — (ii)

$a_2 = 15$ — (iii).

$a_0 = [\alpha_{1,0} + \alpha_{1,1} \cdot 0 \cdot] 1^0 = 1$

$\alpha_{1,0} = 1.$

$[\alpha_{1,0} + \alpha_{1,1} \cdot 1] 1^1 = 9$

$\alpha_{1,0} + \alpha_{1,1} = 9$

$[\alpha_{1,0} \cdot + \alpha_{1,1} \cdot 2] 1^2 + \alpha_{2,0} \cdot 3^2 = 15$

$(\alpha_{1,0} + \alpha_{1,1} \cdot 0 \cdot) 1^0 + \alpha_{2,0} \, 3^0 = 1$

$\alpha_{1,0} + \alpha_{2,0} = 1$ — (i)

$(\alpha_{1,0} + \alpha_{1,1} \cdot 1) 1^1 + \alpha_{2,0} \, 3^1 = 9$

$\alpha_{1,0} + \alpha_{1,1} + 3\alpha_{2,0} = 9$ — (ii)

$(\alpha_{1,0} + \alpha_{1,1} 2) 1^2 + \alpha_{2,0} \, 3^2 = 15$

$\alpha_{1,0} + 2\alpha_{1,1} + 9\alpha_{2,0} = 15$ — (iii) from (ii) & (iii)

$$2\alpha_{1,0} + \cancel{2\alpha_{1,1}} + 6\alpha_{2,0} = 18$$
$$\alpha_{1,0} + \cancel{2\alpha_{1,1}} + 9\alpha_{2,0} = 15$$
$$- \quad - \quad - \quad -$$
$$\alpha_{1,0} - 3\alpha_{2,0} = 3 \quad \text{— (iv)}$$

from (i) & (iv)

$$\cancel{\alpha_{1,0}} - 3\alpha_{2,0} = 3$$
$$\cancel{\alpha_{1,0}} + \alpha_{2,0} = 1$$
$$- \quad - \quad -$$
$$-4\alpha_{2,0} = 2$$
$$\alpha_{2,0} = \frac{-1}{2}$$

$$\therefore \alpha_{1,0} = 1 + \frac{1}{2} = \frac{3}{2}$$

$$\alpha_{1,1} = 9 - 3\alpha_{2,0} - \alpha_{1,0}$$
$$= 9 + \frac{3}{2} - \frac{3}{2}$$

$$= 9$$

The solution of sequence is :-

$\{a_n\}$ is $a_n = \left(\frac{3}{2} + 9 \cdot n\right) 3^n + \left(\frac{-1}{2}\right) 3^n$

For $n = 0$,

$$a_0 = \frac{3}{2} - \frac{1}{2} = 1$$

Exercise:-

Find the solution to the recurrence relation $a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$ with initial conditions $a_0 = 1$, $a_1 = -2$ and $a_2 = -1$.

Solution:-

Characteristic eqn is

$$r^3 + 3r^2 + 3r + 1 = 0$$

~~or, $r^2(3r^2+1)$~~ or, ~~$(r+1)^3 = 0$~~

$$\text{or, } (r+1)(r+1)(r+1) = 0$$

$$\therefore r = -1 \text{ and } m = 3$$

~~$\therefore a_n = (\alpha_{1,0} + \alpha_{1,1} n + \alpha_{2,1} n^2) 1^n$~~

$$\therefore a_n = (\alpha_{1,0} + \alpha_{1,1} n + \ldots + \alpha_{1,m-1} n^{m-1}) r^n$$

$$\text{or, } a_n = (\alpha_{1,0} + \alpha_{1,1} n + \alpha_{1,2} n^2)(-1)^n$$

We have,

$a_0 = 1$

$a_1 = -2$

$a_2 = -1$

$$(\alpha_{1,0} + \alpha_{1,1} \cdot 0 + \alpha_{1,2} \cdot 0^2)(-1)^0 = 1$$

$$\alpha_{1,0} = 1 \quad \text{— (i)}$$

$$(\alpha_{1,0} + \alpha_{1,1} \cdot 1 + \alpha_{1,2} \cdot 1^2)(-1)^1 = -2$$

$$-(\alpha_{1,0} + \alpha_{1,1} + \alpha_{1,2}) = -2 \quad \text{— (ii)}$$

~~$\alpha_{1,0} + \alpha_{1,1} + \alpha_{1,2} = 2$ — (ii)~~

~~From (i),~~ ~~$\alpha_{1,1} + \alpha_{1,2} = -1$ — (ii)~~

$$(\alpha_{1,0} + \alpha_{1,1}\,2 + \alpha_{1,2}\,2^2)(-1)^2 = -1$$

$$\alpha_{1,0} + 2\alpha_{1,1} + 4\alpha_{1,2} = -1. \quad \text{— (iii)}$$

from (ii) & (iii).

$$-\alpha_{1,0} - \alpha_{1,1} - \alpha_{1,2} = -2$$

$$\alpha_{1,0} + 2\alpha_{1,1} + 4\alpha_{1,2} = -1$$

$$\alpha_{1,1} + 3\alpha_{1,2} = -3$$

from (i) & (ii)

$$-1 - \alpha_{1,1} - \alpha_{1,2} = -2$$

$$-\alpha_{1,1} - \alpha_{1,2} = -1$$

$$\alpha_{1,1} + \alpha_{1,2} = 1$$

or, $\alpha_{1,1} = 1 - \alpha_{1,2}$.

Now, in (iii)

$$1 + 2 - 2\alpha_{1,2} + 4\alpha_{1,2} = -1$$

$$1 + 2 + 2\alpha_{1,2} = -1$$

$$3 + 2\alpha_{1,2} = -1.$$

$$2\alpha_{1,2} = -4$$

$$\alpha_{1,2} = -2$$

Or, $\alpha_{1,1} = 1 - (-2) = 3.$

∴ The sol^n of sequence $\{a_n\}$ is

$$a_n = [1 + 3n + (-2)n^2](-1)^n.$$

2. Solving linear Non-homogenous Recurrence Relation of degree K with constant coefficients.

The recurrence relation of the form
$a_n = C_1 a_{n-1} + C_2 a_{n-2} + \dots + C_k a_{n-k} + F(n)$ where $c_1, c_2, \dots c_n$ are real numbers and $F(n)$ is a function depending upon n. The recurrence relation preceding $F(n)$ is called associated homogenous recurrence relation.

Theorem 5:-

If $\{a_n^{(P)}\}$ is a particular solution of the non homogenous linear recurrence relation with constant coefficient $a_n = C_1 a_{n-1} + C_2 a_{n-2} + \dots + C_k a_{n-k} + F_n$ then every solution of the form $\{a_n^{(P)} + a_n^{(n)}\}$ where $a_n^{(n)}$ is a solution of a associated homogenous recurrence relation,

$a_n = C_1 a_{n-1} + C_2 a_{n-k} + \dots a_k a_{n-k}$.

Example:-

Find all the solution of the recurrence relation $a_n = 4a_{n-1} + n^2$. Also find the solution of relation with initial condition $a_1 = 1$

Solution,

here the associated recurrence relation is
$a_n = 4a_{n-1}$.

Characteristic eqn:

$\cancel{r^2 - 4r + 1 = 0}$
$\cancel{r^2 - 2r - 2r + 1 = 0}$

$r - 4 = 0$
$r = 4$.
$\therefore a_n = \alpha (4)^n$

$\{a_n^{(h)}\} = \alpha 4^n$.

Now,

$F(n) = n^2$ is a polynomial of degree 2, a trial solution is a quadratic function in n, say $p_n = an^2+bn+c$ where a, b & c are constants.

To determine whether there are any solutions of the form, suppose that $p_n = an^2+bn+c$ is such solution. Then the eq^n is $a_n = 4a_{n-1} + n^2$ becomes.

$$an^2+bn+c = 4(a(n-1)^2 + b(n-1) + c) + n^2$$
$$\{4a(n-1)^2 + 4b(n-1) + 4c\} + n^2$$
$$= [4a(n^2-2n+1) + 4b(n-1) + 4c] + n^2$$
$$\cancel{= (4an^2 - 4a + 4bn - 4b + 4c) + n^2}$$
$$\cancel{= (4a+1)n^2 + (4b-1)n =}$$
$$= (4a+1)n^2 + (-8a+4b)n + (4a-4b+4c)$$

an^2+bn+c is a sol^n, if

$$\cancel{a = 4a+1 \Rightarrow 5a = -1/4}$$
$$\cancel{b = -8a+4b \Rightarrow 2+4b \Rightarrow b = 1/2}$$
$$\cancel{c = 4a-4b+4c \Rightarrow -1-2+4c \Rightarrow c = 3/4}$$
$$a = 4a+1 \therefore a = -1/3$$
$$b = -8a+4b \therefore b = -8/9$$
$$c = 4a-4b+4c \therefore c = -20/27$$

$$\therefore \{a_n^{(p)}\} = an^2+bn+c = -\frac{1}{3}n^2 - \frac{8}{9}n - \frac{20}{27}$$

Now,

the solution of $\{a_n\}$ is

$$\{a_n^{(p)} + a_n^{(h)}\} =$$
$$\therefore a_n = \alpha 4^n - \frac{1}{3}n^2 - \frac{8}{9}n - \frac{20}{27}$$

Now, $a_1 = 1$.

$$a_n = \alpha(4)^n + \left(-\frac{1}{3}n^2 - \frac{8}{9}n - \frac{20}{27}\right)$$

$$\alpha(4)^1 + \left(-\frac{1}{3}\cdot 1^2 - \frac{8}{9}\cdot 1 - \frac{20}{27}\right) = 1$$

$$4\alpha + \left(-\frac{53}{27}\right) = 1$$

$$\cancel{4\alpha} = \frac{\cancel{53}}{\cancel{27}}$$

$$\alpha = \frac{20}{27}$$

$\therefore$ The final solution of $\{a_n\}$

$$a_n = \frac{20}{27}(4)^n + (an^2 + bn + c)$$

Exercise :-

Find all the solutions of the recurrence relation $a_n = 3a_{n-1} + 2n$. What is the solution with $a_1 = 3$.

Solution:

here the associated recurrence relation is

$$a_n = 3a_{n-1}$$

Characteristic eqⁿ :-

$$r - 3 = 0$$
$$r = 3$$
$$\therefore a_n = \alpha(3)^n$$
$$\{a_n^{(h)}\} = \alpha 3^n.$$

Now, $f(n) = 2n$ is a polynomial of degree 1, a trial solution of a linear function is n, say $p_n = an + c$ where a & c are constants.

To determine whether there are solution of the form, suppose $p_n = an + c$ is such solⁿ. Then eqⁿ is $a_n = 3a_{n-1} + 2n$ becomes

$$an + c = 3[a(n-1) + c] + 2n$$
$$= 3[an - a + c] + 2n$$
$$= 3an + 3c + 2n - 3a.$$
$$= n(3a + 2) + (-3a + 3c)$$
$$= (3a + 2)n + (-3a + 3c)$$

$\therefore$ $an + c$ is solⁿ if

$a = 3a + 2$ or, $2a = -2 \Rightarrow a = -1$

$c = 3c - 3a$ or, $c = 3c + 3$

or, $3c - c = -3$ or, $c = -3/2$

$\therefore \{a_n^{(p_n)}\} = a_n + c = -n - \frac{3}{2}$

Now,

the solution of $\{a_n\}$ is

$$\{a_n^{(p)} + a_n^{(h)}\}$$

$\therefore a_n = \alpha 3^n - n - \frac{3}{2}$

Now,

$a_1 = 3$

$a_n = \alpha 3^n - n - \frac{3}{2}$

or, $3 = \alpha 3^1 - 1 - \frac{3}{2}$

or, $3 = \alpha 3 - \frac{2-3}{2}$

or, $3 = 3\alpha - \frac{5}{2}$

or, $3\alpha = 3 - \frac{5}{2}$

or, $3\alpha = \frac{11}{2}$

$\therefore \alpha = \frac{11}{6}$

$\therefore$ Final solution of $\{a_n\}$ is

$$a_n = \frac{11}{6}(3)^n - n - \frac{3}{2}$$

Theorem 6:-

Suppose that $\{a_n\}$ satisfies the linear ~~le~~ non-homogenous recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k} + F(n)$, where $c_1, c_2 \dots c_k$ are real numbers and $F(n) = (b_t n^t + b_{t-1} n^{t-1} + \dots b_1 n + b_0) s^n$, where $b_0, b_1 \dots b_t$ and s are real numbers.

When s is not a root of the characteristic eq^n of the associated linear homogenous recurrence relation, there is a particular solution of the form
$$(P_t n^t + P_{t-1} n^{t-1} + \dots + P_1 n + P_0) s^n.$$

When s is a root of the characteristic eq^n and its multiplicity is m, there is a particular solution of the form $n^m (P_t n^t + P_{t-1} n^{t-1} + \dots + P_1 n + P_0) s^n$.

Example:-

Find a solution of the recurrence relation $a_n = 2a_{n-1} + n\,2^n$.

Sol^n:-

here the associative linear homogenous recurrence relation is
$$a_n = 2a_{n-1}$$

Characteristic eq^n is
$$r - 2 = 0$$
$$r = 2.$$

$\therefore$ Sol^n of $\{a_n\}$ is $\alpha 2^n$

$$\{a_n\} = n^m (P_t n^t + P_t n^{t-1} \dots P_0) s^n$$
$$= n^1 (p_1 n^1 + P_0 n^0) 2^n$$
$$= n (p_1 n + p_0) 2^n$$

$F(n) = n \cdot 2^n$

$\therefore$ The function is $p = (b_1 n + b_0) S^n$

Find the ~~s~~^{n} of the recurrence relation ~~$a_n = 6a_{n-1}$~~

$$a_n = 6a_{n-1}\ 9a_{n-2} + n3^n.$$

Find the S' of the recurrence relation

$a_n = 6a_{n-1} - 9a_{n-2} + 3^n$

Counting

Introduction

→ ~~Combination~~ Combinatorics is the study of arrangement of objects.

→ Enumeration the counting of object with certain properties is an important part of combinatorics.

→ we must count objects to determine the complexity the algorithm, to determine there are enough telephone number to meet demand etc.

Basis of Counting

Sum Rule:

If a task can be done in n_1 ways and a second task in n_2 ways and if these tasks cannot be done at same time, then there are n_1+n_2 ways to done one of these tasks.

i.e. $|A_1 \cup A_2 \cup A_3 \cup \dots \cup A_m| = |A_1|+|A_2|+ \dots + |A_m|$.

Example:-

In how many ways we can draw a heart or a diamond from an ordinary deck of playing cards?

Soln:-

The number of ways to draw diamond $(n_1) = 13$

The number of ways to draw heart $(n_2) = 13$.

$\therefore$ The total number of ways $(n_1+n_2) = 26$

Example:-

In how many ways we can get a sum of 4 or of 8 when two distinguishable dice are rolled?

Solution:- By sum rule.

No. of ways we can get a sum of 4 is $(n_1) = 3$

No. of ways we can get a sum of 8 is $(n_2) = 5$

$\therefore$ The total number of ways $(n_1+n_2) = 8$.

Product Rule

If a task can be done in n_1 ways and a second task in n_2 ways, after the first task has been done, then there are $n_1 n_2$ ways to do the work that consists both the task.

i.e. $|A_1 \times A_2 \times \ldots A_k| = |A_1| \cdot |A_2| \ldots |A_k|$

Example:-

An office building contains 27 floors and has 37 offices on each floor. How many offices are there in the building?

Solution:-

By Product Rule;

no. of floors $(n_1) = 27$

no. of offices $(n_2) = 37$.

Total no. of offices $= n_1 n_2 = 27 \times 37$

How many different three-letter initials with none of the letters can be repeated can people have?

Solution:-

The different ways ways in which three-letter initials with none of the letters can be repeated can people have $= 26 \times 25 \times 24$.

The inclusion-exclusion Principle:-

When two tasks can be done at the same time, we cannot use the sum rule to count the number of ways to do one of the two tasks. Adding the number of ways to do each task leads to an over count, since the ways to do both tasks are counted twice. To correctly count the number of ways to do one of the two tasks, we add the number of ways to do both tasks. This technique is called principle of inclusion-exclusion.

Let A_1 and A_2 be sets and let T_1 be the task of choosing element from A_1 and T_2 be the task of choosing an element from A_2. The number of ways to do either T_1 or T_2 is.

$$|A_1 \cup A_2| = |A_1| + |A_2| = |A_1 \cap A_2|$$

Exercise:-
How many bit strings of length ~~eight~~ eight either start with a 1 bit or end with two bits 00?

Solution:-

A_1 = bits starting with 1
A_2 = bits ~~starting~~ ending with 00

$A_1 = 2^7$ ways : 1 _ _ _ _ _ _ _
$A_2 = 2^6$ ways = _ _ _ _ _ _ 0 0

$|A_1 \cap A_2|$ = 1 _ _ _ _ _ 0 0 $= 2^5$ ways.

By principle of inclusive exclusive principle.

$\therefore |A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$
$= 2^7 + 2^6 - 2^5$
= 160 ways.

Pigeonhole Principle:-

The pigeonhole Principle (Theorem):-
If K+1 or more objects are placed into K boxes then there is at least one box containing two or more of the objects.

Proof:-
We use proof by contradiction. Suppose that K+1 or more boxes are placed into K boxes and no boxes such that there are no two objects in a box. This contradicts our assumption. So, there is at least one box assuming two or

more of the objects.

Example:-

Show that if there are 30 students in a class, then at least two have last name that begins with same letter.

Solution:-

total number of students = 30

number of alphabets = 26.

By pigeonhole principle, there are atleast two students having last name that begins with same letter.

Example:-

How many students must be in a class to guarantee that at least two students recieve the same score on the final exam, if the exam is graded on a scale from 0 to 100 points?

Solution:-

n = no. of students in class.

score in exam = 0 to 100

By pigeonhole principle,

~~Atleast~~ There must be at least 102 students.

The genesalised pigeonhole principle

If N objects are placed into K boxes there more is at least $[N/K]$ objects.

Proof:-
Suppose one of the boxes contain $\lceil N/K \rceil$ or more objects. Then every box contain at most $\lceil N/K \rceil - 1$ objects. So the total number of objects is at most $K(\lceil N/K \rceil - 1)$.

But $\lceil N/K \rceil - 1 < N/K$

Thus, the total number of object is less than $K(N/K)$ is less than N. This is a contradiction. Hence the prof:

Examples:-
Find the minimum number of people among 100 people who were born in the same month

Solution:-

$[N]$ total number of people $= 100$
no. of month $[K] = 12$

By generalised pigeonhole principle,

$$\lceil N/K \rceil = \lceil 8.33 \rceil$$
$$= 9.$$

If a class has 24 students, what is the max^m number of possible grading that must be done to ensure that there are atleast two students with the same grade.

Solution:-

$N = 24$

$K = ?$

$\lceil N/K \rceil = 2$

We know that, By generalised pigeonhole principle,

$N = K(r-1)+1$

or, $24 = K(2-1)+1$

or, $24 = K+1$

$\therefore K = 23$,

What is the minimum number of students required in a class to be sure that at least six will receive the same grade, If there are five possible grades A, B, C, D, E & F?

Solution!-

$N = ?$

$K = 5$

$N/K = 6$.

Acc. to G.P.P.

$N = K[N/K - 1] + 1$

$= 5[6-1]+1$

$= 26$

Permutations and Combinations:-

Permutations.

A permutation of a set of distinct objects is an ordered arrangement of these objects. An ordered arrangement of r- elements of a set is called r- permutation.

$$\therefore P(n,r) = \frac{n!}{(n-r)!} \quad \text{and} \quad P(n,n) = n!$$

Example:-

How many ways are there are there to select a first-prize winner, a second prize winner and a third-prize winner from 100 different people who have entered a contest?

Solution:-

$n = 100$

$r = 3$

$$P(n,r) = \frac{n!}{(n-r)!} = \frac{100!}{(100-3)!} = \frac{100!}{97!}$$

How many permutation of the letters ABCDEFGH contains the string ABC.

Solution:-

$n = 6$

$r = 6.$

$$P(n,n) = 6! = 720$$

Suppose that a ~~salesman~~ saleswoman has to visit eight different ~~houses~~ cities. She must begin her trip in a specified city, but she can visit other seven cities in any order she wishes. ~~Ho~~ How many possible orders can ~~be~~ the saleswoman use when visiting these cities?

Solution:-

$n = 7$

$r = 7$

$P(n, n) = 7!$

2. Show by mathematical induction that

$$\sum_{j=0}^{n} ar^j = a + ar + ar^2 + \cdots + ar^n = \frac{ar^{n+1} - a}{r-1} \text{ where}$$

$r \neq 1$. and n is a non-negative integer.

Combinations:-

An r-combination of elements of a set is an unordered selection of r elements from the set.

$$C(n,r) = \frac{n!}{r!(n-r)!}$$

Example:-

How many ways are there to select five players from a 10-member tennis team to make a trip to a match at another college?

Solution

total no. of member(n) = 10

r = 5

$$\therefore C(10,5) = \frac{10!}{5!(10-5)!}$$

$$= 252$$

∴ 252 ways are there to select five players from a 10-member tennis team.

Binomial Coefficients

Theorem:-

let x and y be variables, and let n be non-negative integer. Then,

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$$

$$= \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y^1 + \dots \binom{n}{n-1} x^1 y^{n-1} + \binom{n}{n} y^n$$

1. What is the expansion of $(x+y)^4$?

$$(x+y)^4 = \sum_{j=0} \binom{n}{j} x^{4-j} y^j$$

$$= \binom{4}{0} x^4 + \binom{4}{1} x^3 y^1 + \binom{4}{2} x^2 y^2 + \binom{4}{3} x y^3 + \binom{4}{4} y^4$$

$$= x^4 + 4x^3 y + 6x^2 y^2 + 4xy^3 + y^4$$

2. What is the coefficient of $x^{12} y^{13}$ in the expansion of $(x+y)^{25}$?

Solution:

The coefficient of $x^{12} y^{13}$ is:

$j = 13$

$n - j = 12$

or, $n = 12 + 13 = 25.$

$$\therefore \binom{n}{j} = \binom{25}{13} = 5200300$$

classmate

Date
Page

3. What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x - 3y)^{25}$?

Solution:-

$$((2x)+(-3y))^{25}$$

The coefficient of $x^{12}y^{13}$ is:-

$$j = 13$$

$$\therefore \binom{n}{j} = \binom{25}{13} = -3.39 \times 10^{16}$$

$$= \binom{25}{13}(2)^{12}x^{12}(-3)^{13}y^{13} = -3.39 \times 10^{16}$$

Corollary 1:-

let n be a non-negative integer, Then,

$$\sum_{k=0}^{n}\binom{n}{k} = 2^n$$

Proof:-

Using the binomial theorem with $x=1$ and $y=1$.

$$2^n = (x+y)^n = (1+1)^n$$

Then the binomial expansion will be

$$(1+1)^n = \sum_{k=0}^{n}\binom{n}{k}1^{n-k}1^k = \sum_{k=0}^{n}\binom{n}{k}$$

proved!

Corollary 2:-

let n be a non-negative integer. Then

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$$

Proof:-

let $x=1$, $y=(-1)$.

$0^n = (1+(-1))^n$

$\therefore$ Using binomial theorem,

$$\sum_{k=0}^{n} \binom{n}{k} (1)^{n-k} (-1)^k = \sum_{k=0}^{n} \binom{n}{k} \times 0$$

$$= 0$$

proved!

Corollary 3: Let n be a non-negative integer. Then,

$$\sum_{k=0}^{n} (2)^k \binom{n}{k} = 3^n.$$

Proof:-

Using binomal theorem with $x=1$ & $y=2$.

$\therefore \cancel{\sum_{k=0}^{n} \binom{n}{k} (1)^{n-k} (2)^k} =$

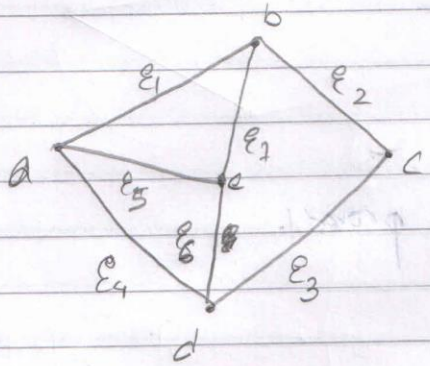
$3^n = (1+2)^n$

$$(1+2)^n = \sum_{k=0}^{n} \binom{n}{k} (1)^{n-k} (2)^k$$

$$= 2^k \sum_{k=0}^{n} \binom{n}{k}$$

proved!

Unit - 4: Graphs



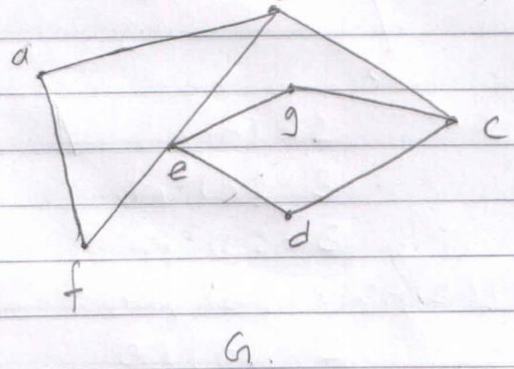
$G(V, E)$

$E = \{(a, b), (b, c), (c, d), (d, a), (a, e), (b, e), (d, e)\}$

$V = \{a, b, c, d, e\}$

Types of Graph

1. Simple Graph.



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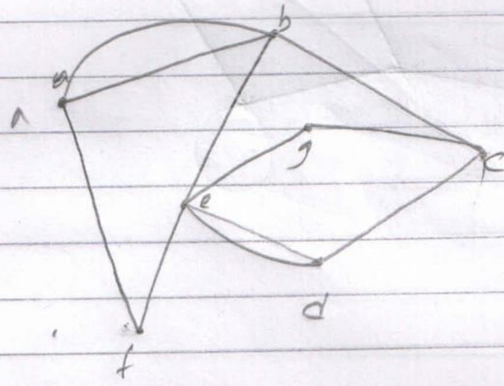
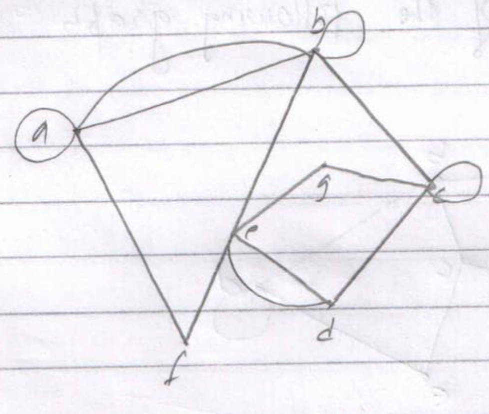
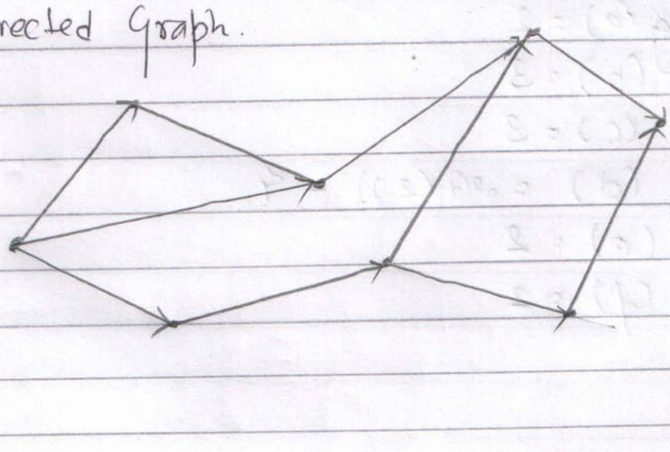
2. Multigraph



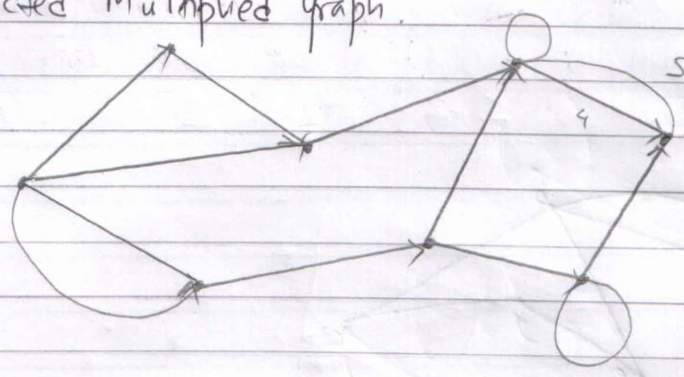
3. Pseudograph.



3. Directed Graph.



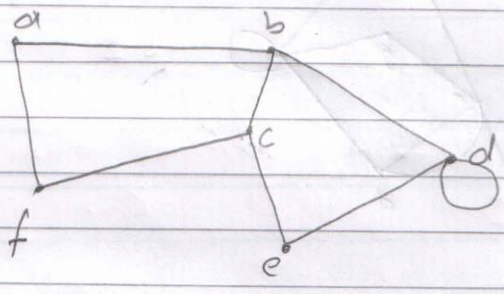
4. Directed Multiplied Graph.



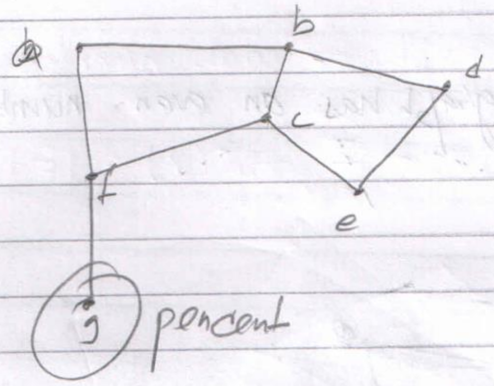
Graph Terminologies.

Degree of vertices:-

Find the degree of the following graph.



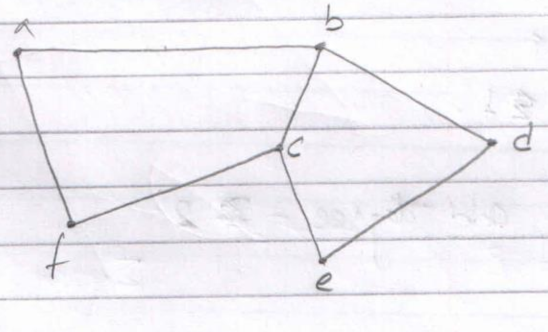
deg (a) = 2
deg (b) = 3
deg (c) = 3
deg (d) = 2(+(2)) = 4.
deg (e) = 2
deg (f) = 2



Theorem: The Handshaking theorem

Let $G = (V, E)$ be an undirected graph with e edges.
Then,

$$2e = \sum_{v \in V} \deg(v)$$

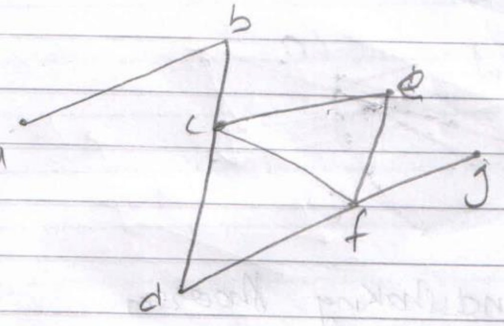


$\deg(a) = 2$
$\deg(b) = 3$
$\deg(c) = 3$
$\deg(d) = (2)$
$\deg(e) = 2$
$\deg(f) = 2$.

Total deg $= 14$
$e = 7$
$\therefore \deg = 2e$

Theorem :

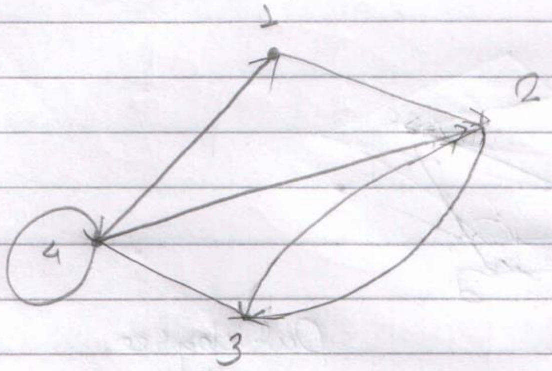
An undirected graph has an even number of vertices of odd degree.



deg (a) = 1
deg (b) = 2
deg (c) = 4
deg (d) = 2
deg (e) = 2
deg (f) = 4
deg (g) = ~~2~~ 1

vertices with odd degree = ~~4~~ 2

Find the in-degree and out-degree in the following graph.



In-degree $deg^{-}(a)$ | Out-degree $deg^{+}(a)$ ← Representation

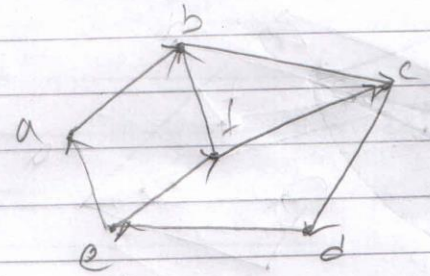
In-degree	Out degree
$deg^{-}(1) = 1$	$deg^{+}(1) = 1$
$deg^{-}(2) = 3$	$deg^{+}(2) = 1$
$deg^{-}(3) = 2$	$deg^{+}(3) = 1$
$deg^{-}(4) = 1$	$deg^{+}(4) = 3$

Theorem:-

$G(V, E)$ be a graph ~~the~~ with directed ~~origin~~ edges.

Then,

$$\sum_{v \in V} \deg^-(v) = \sum_{v \in V} \deg^+(v) = |E|$$



In degree	Out degree
~~$\deg^-(a) = 1$~~	~~$\deg^+($~~ —
$\deg^-(a) = 1$	$\deg^+(a) = 1$
$\deg^-(b) = 2$	$\deg^+(b) = 1$
$\deg^-(c) = 2$	$\deg^+(c) = 1$
$\deg^-(d) = 1$	$\deg^+(d) = 1$
$\deg^-(e) = 2$	$\deg^+(e) = 1$
$\deg^-(f) = 1$	$\deg^+(f) = 2$

$\therefore \sum \deg^-(v) = 8$

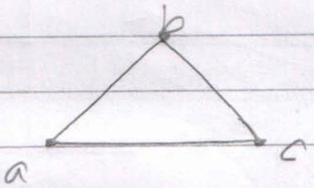
$\sum \deg^+(v)$

Complete graph

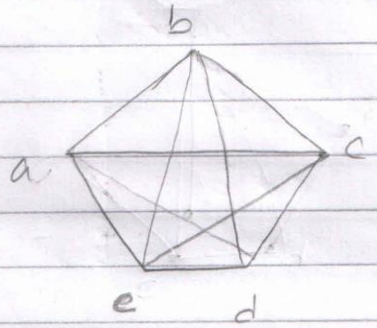
K_n

K_1

K_3



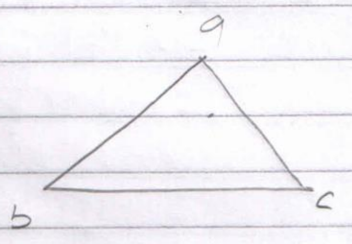
K_5



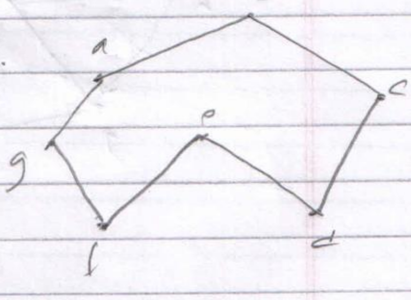
Cycles:-

C_n

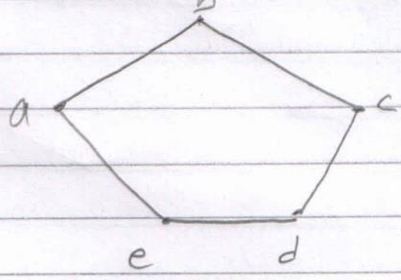
C_3



C_7:



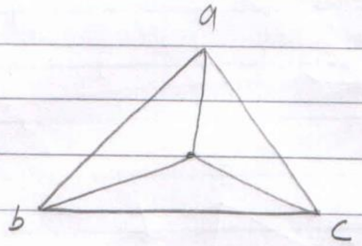
C_5



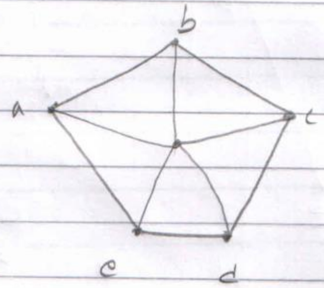
Wheels :-

W_n

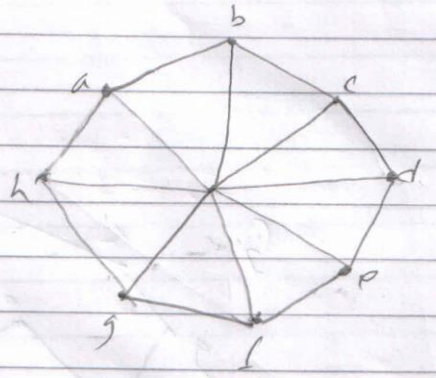
W_3.



W_5



W_7.

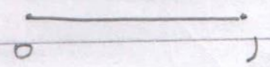


n - Cubes :-

$Q_n \to 2^n$

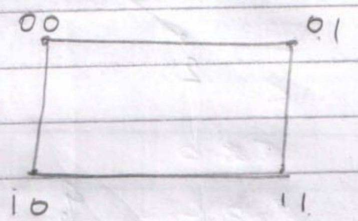
Q_1

no. of vertices $= 2^1 = 2$



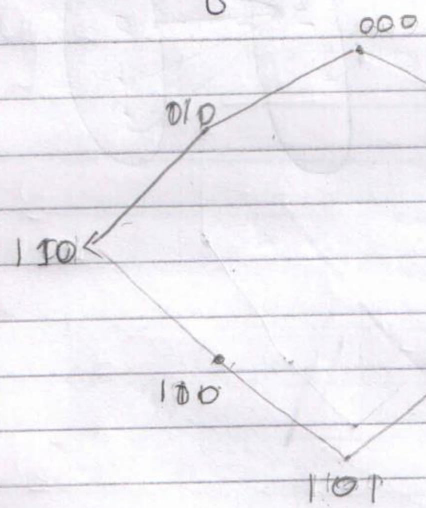
Q_2

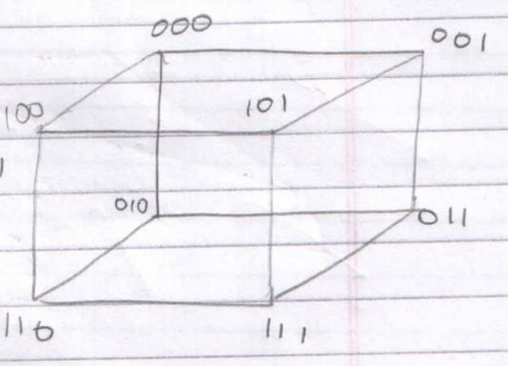
no. of vertices $= 2^2 = 4$



Q_3

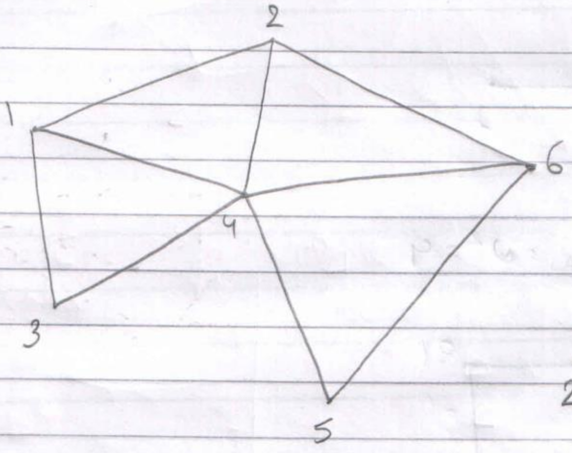
no. of vertices $= 2^3 = 8$



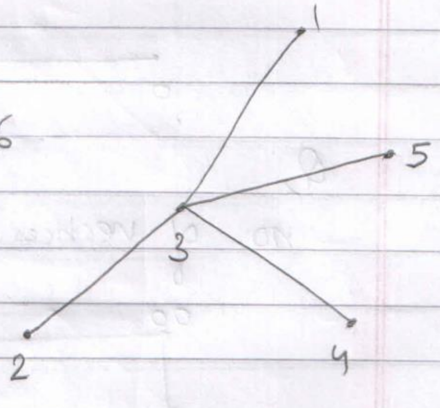


Biparhile Graph.

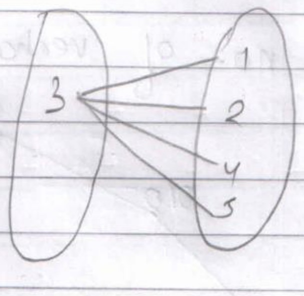
Are the graph below biparate?



G_1



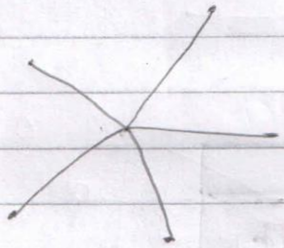
G_2

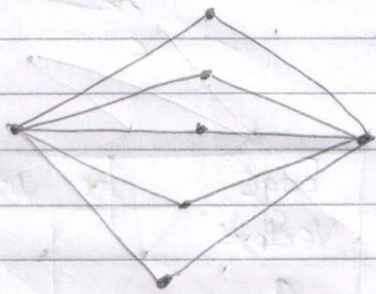


Complete Biparhle Graph.

$K_{m,n}$

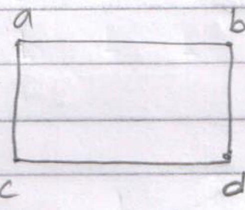
$K_{1,5}$. $K_{2,5}$.



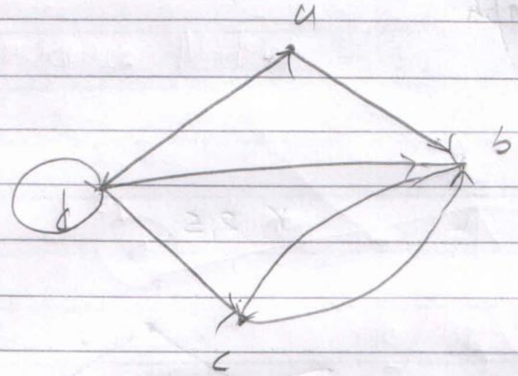


Graph Representation :-

1. Adjacency List :-

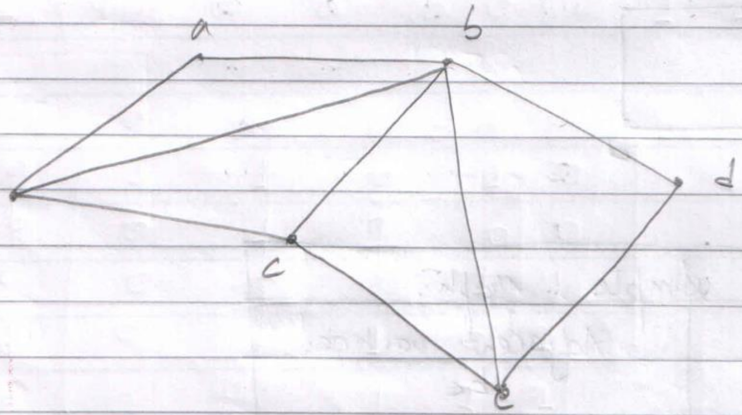


Edge for simple graph.	
Vertex	Adjacent vertices
a	bc
b	ad
c	ad
d	bc

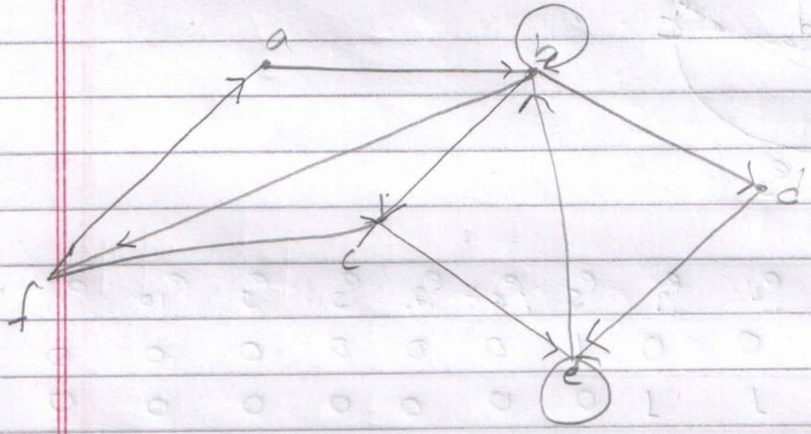


Edge for simple graph	
Vertex	Adjacent graph
a	b
b	c
c	b
d	a, b, c

2. Adjacency Matrix :-

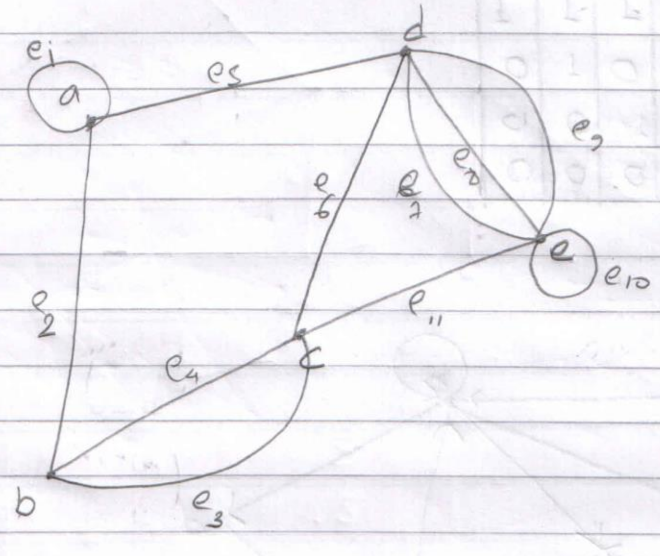


	a	b	c	d	e	f
a	0	1	0	0	0	1
b	1	0	1	1	1	1
c	0	1	0	1	1	1
d	0	1	0	0	1	0
e	0	1	1	1	0	0
f	1	1	1	0	0	0



$$
\begin{array}{c|c}
 & \begin{matrix} a & b & c & d & e & f \end{matrix} \\
\begin{matrix} a \\ b \\ c \\ d \\ e \\ f \end{matrix} &
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 0 & 0
\end{bmatrix}
\end{array}
$$

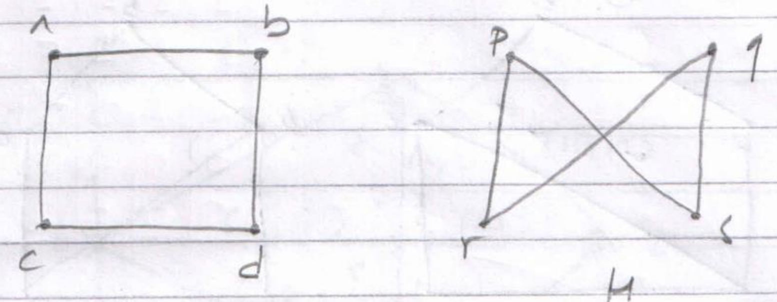
3. Incidence Matrix.



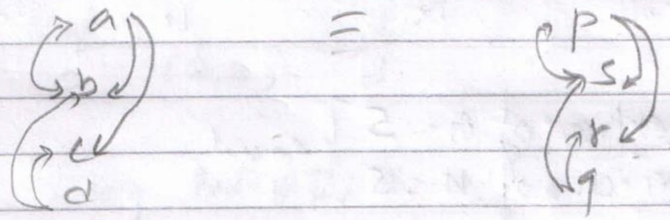
$$
\begin{array}{c|ccccccccccc}
 & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9 & e_{10} & e_{11} \\
\hline
a & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
b & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
c & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
d & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
e & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1
\end{array}
$$

$$
\begin{array}{c|ccccc}
 & a & b & c & d & e \\
\hline
a & 1 & 1 & 0 & 1 & 0 \\
b & 0 & 1 & 2 & 0 & 0 \\
c & 0 & 2 & 0 & 1 & 1 \\
d & 0 & 0 & 1 & 0 & 3 \\
e & 0 & 0 & 1 & 3 & 1
\end{array}
$$

Isomorphic Graph :-

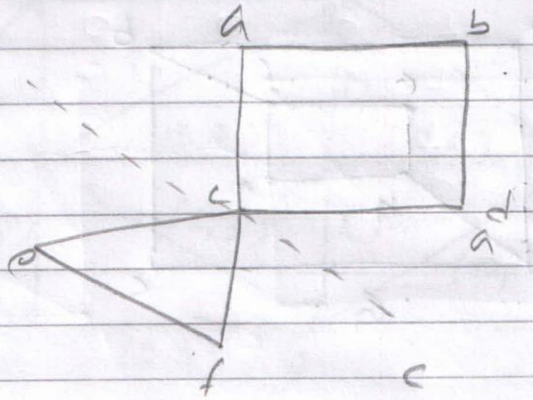


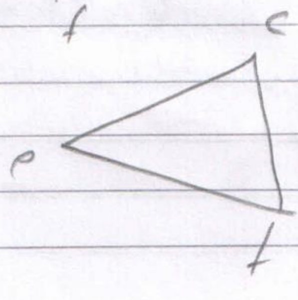
G H



Invariant Rule

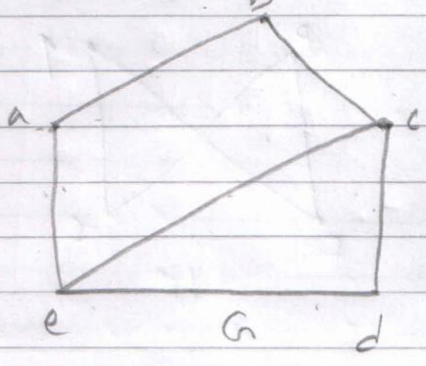
1. no. of vertices must be equal.
2. no. of edges must be equal.
3. no. of degree of vertices must be equal.
4. subgraph consisting of same degree must be same.
5. Adjacent matrix must be same.



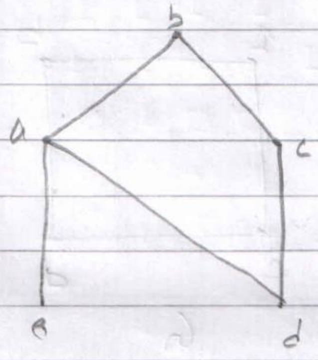


⇒ subgraph of G

Show that the graph displayed below are not isomorphic.



G



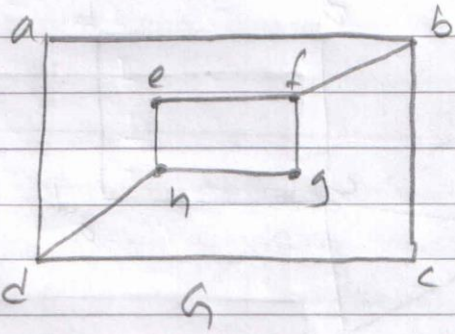
H

no. of vertices of $G = 5$
no. of vertices of $H = 5$ } equal.

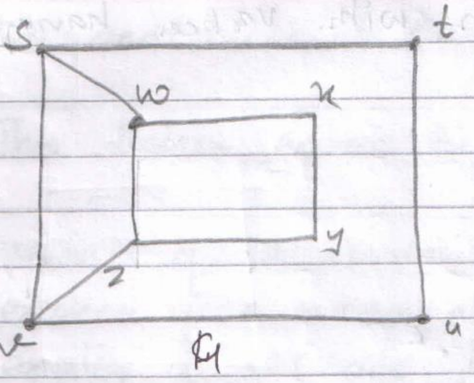
no. of edges of $G \neq$ no. of edges of H
i.e, $6 \neq 5$.

Hence graphs are not isomorphic.

Determine whether the graphs shown below are isomorphic or not.



G



1. No. of vertices of G ~~is~~ = No. of vertices of H = 8.

2. No. of edges of G = No. of edges of H = 10.

3. Degree of vertices of G

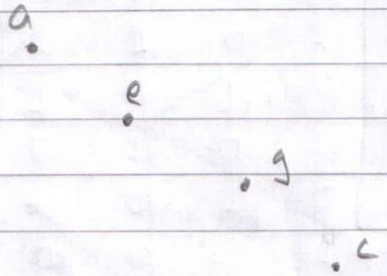
deg (a) = 2
deg (b) = 3
deg (c) = 2
deg (d) = 3
deg (e) = 2
deg (f) = 3
deg (g) = 2
deg (h) = 3

Degree of vertices of H.

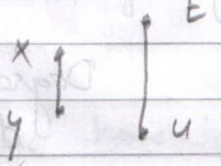
deg(s) = 3
deg(t) = 2
deg(u) = 2
deg(v) = 3
deg (w) = 3
deg (x) = 2
deg (y) = 2
deg (z) = 3

degree of Vertices of ~~G~~ G = degree of vertices of H as there are equal no. of vertices having degree 2 & 3.

Subgraph of G with vertices having degree 2.

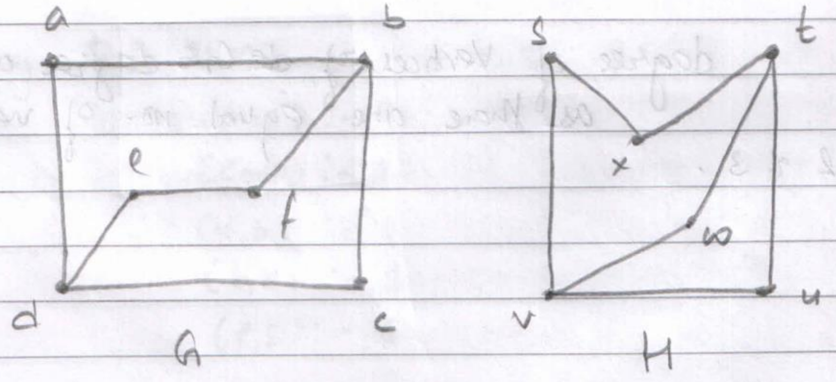


Subgraph of H with vertices having degree 2 is,



Hence subgraph are not same they are not isomorphic.

Determine whether the graph shown below are isomorphic or not.



1 no. of vertices of G = no. of vertices of H. = 7

2 no. of edges of G = no. of edges of H = 6

3 degree of vertices of G.

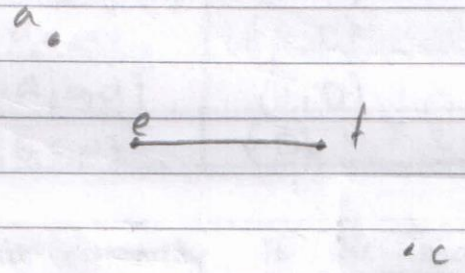
deg (a) = 2
deg (b) = 3
deg (c) = 2
deg (d) = 3
deg (e) = 2
deg (f) = 2

degree of vertices of H

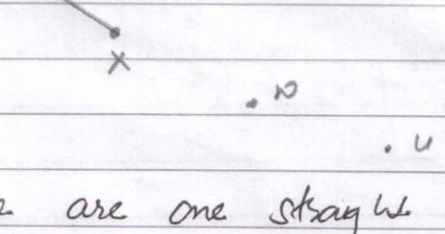
deg (s) = 2
deg (t) = 3
deg (u) = 2
deg (v) = 3
deg (w) = 2
deg (x) = 2

degree of vertices of G = degree of vertices of H as there as equal no. of ~~degree~~ vertices having g degree 2 & 3.

4. subgraph of G with vertices of degree 2.



subgraph of H with vertices of degree 2.



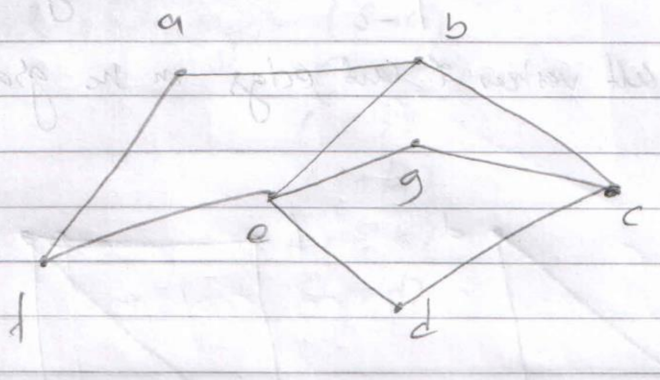
since these are one straight line and 2 vertices common they are equal.

5. Adjacency matrix of G.

$$
\begin{array}{c|cccccc}
 & a & b & c & d & e & f \\
\hline
a & 0 & 1 & 0 & 1 & 0 & 0 \\
b & 1 & 0 & 1 & 0 & 0 & 1 \\
c & 0 & 1 & 0 & 1 & 0 & 0 \\
d & 1 & 0 & 1 & 0 & 1 & 0 \\
e & 0 & 0 & 0 & 1 & 0 & 1 \\
f & 0 & 1 & 0 & 0 & 1 & 0
\end{array}
$$

Adjacency matrix of H

Graph Connectivity



From the graph given above show the paths and circuit of length 4

⇒ Path:

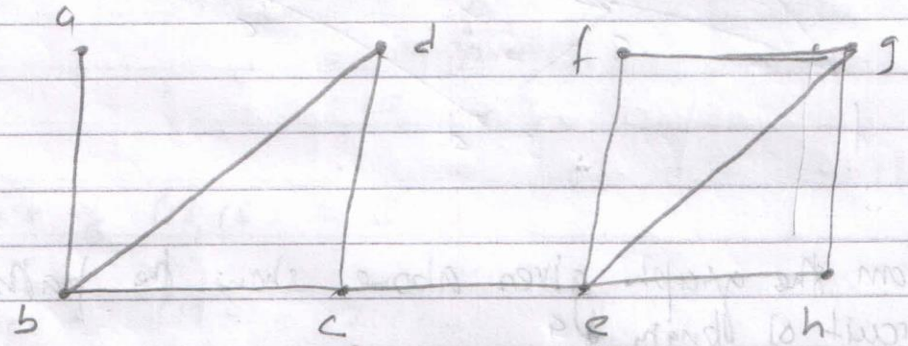
a - b - c - g - e
a - f - e - g - c
a - b - e - g - c
b - c - d - e - f
a - f - e - d - c
b - c - g - e - f
b - e - d - c - g
f - e - g - c - b
and so on

Circuit:

a - b - e - f - a
b - e - g - c - b
e - g - c - d - e
e - b - c - d - e
~~and so on~~ and so on

Cut vertices (articulation points) and cut edge (bridge)

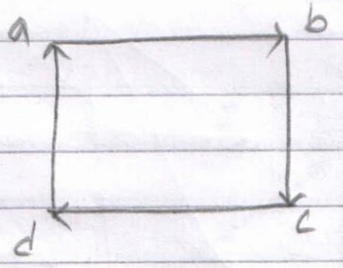
Find out the cut vertices & cut edges in the graph given below:



cut vertices = {c, e, b}

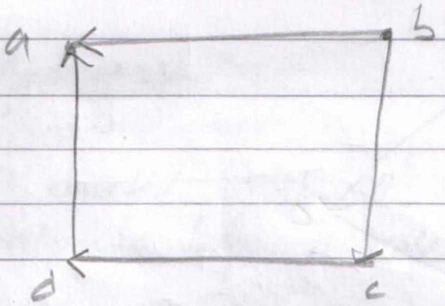
cut edges = {(c,e), (a,b)}

Connectedness in directed graphs.

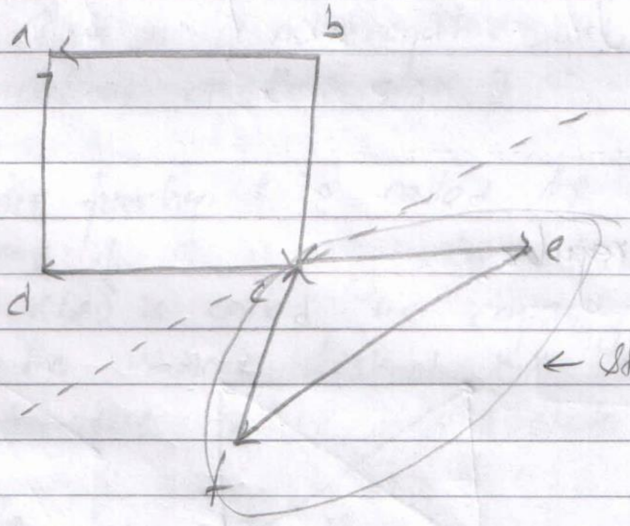


~~every~~ every vertex having path to go to every vertex

strongly connected.

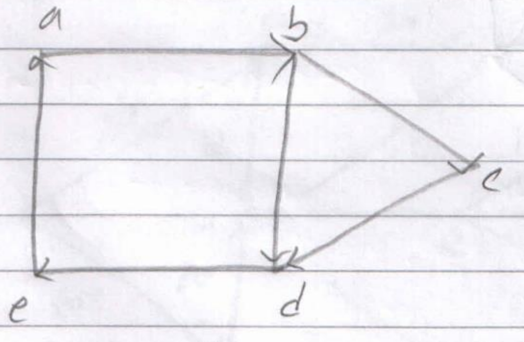


(weakly connected)

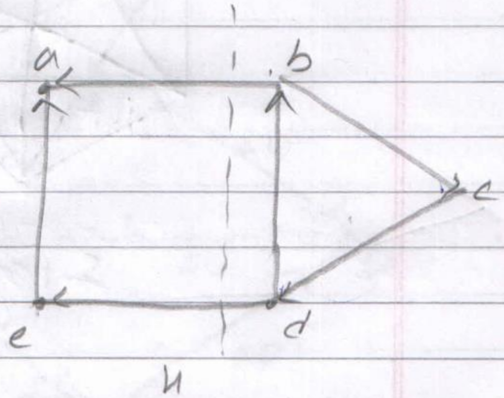


(weakly connected).

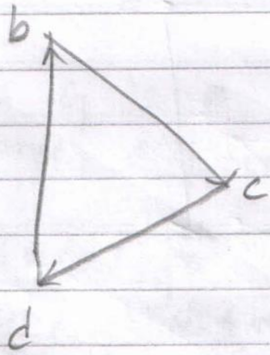
Are the directed graphs shown below strongly connected? Are they weakly connected? Determine the strongly connected component if any.



(contsoly)



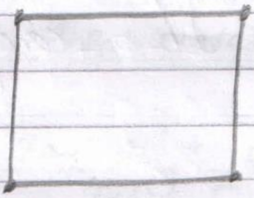
(weakly)



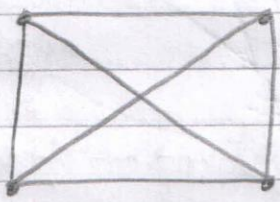
strongly connected component of G

Regularly Graph.

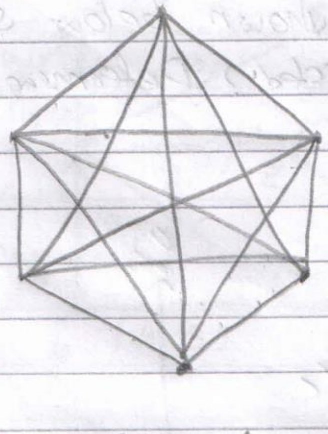
<u>n</u>- regular.



2-regular



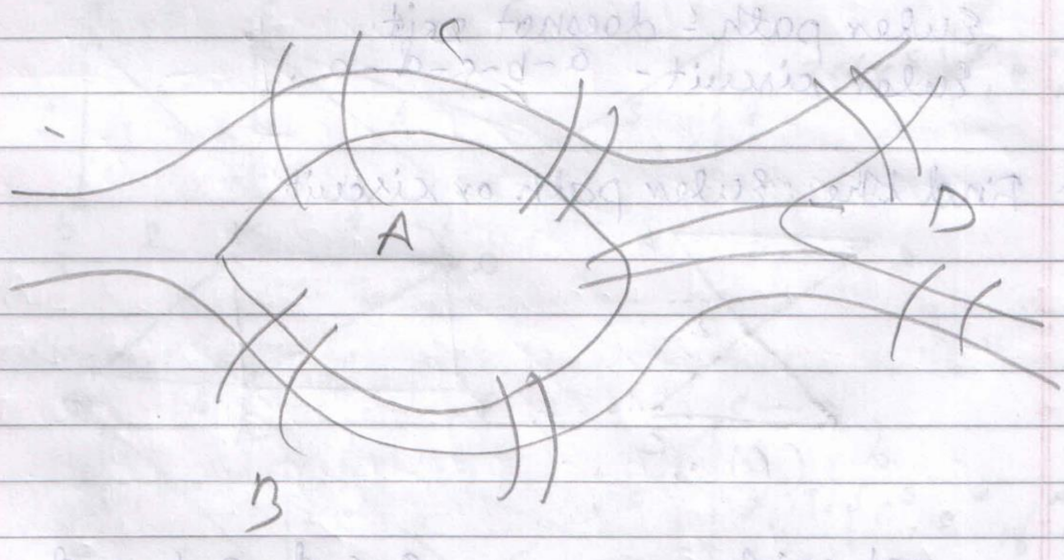
3- regular

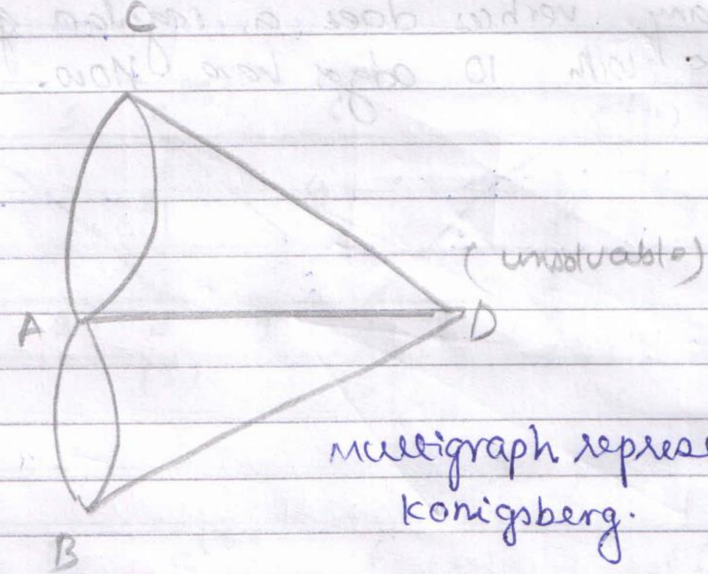


5-regular

How many vertices does a regular graph of degree 4 with 10 edges have? How?

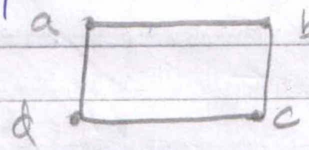
Euler paths & circuits.





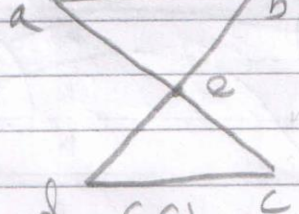
multigraph representing konigsberg.

Euler path and circuit

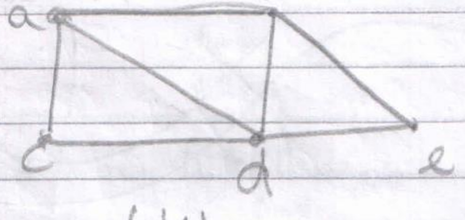


Euler path - doesnot exist.
Euler circuit - a-b-c-d-a

4. Find the Euler path or circuit



(a)



(b)

	(a)	(b)
Euler path:	not exist	a-c-d-a-b-e-d
Euler circuit:	a-b-e-d-c-e-a	no exist.

Necessary and sufficient condition for Euler circuits and paths.

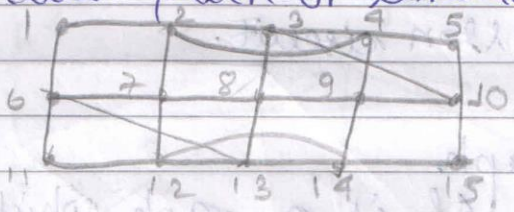
Theorem: A connected multigraph has an Euler circuit if and only if each of its vertices have even degree.

necessary condition :- A connected multigraph has an Euler circuit if degree of every vertex is even.

Theorem:
A connected multigraph has an Euler path but not circuit if and only if it has exactly two vertices of odd degree.

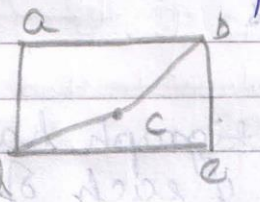
necessary sufficient:

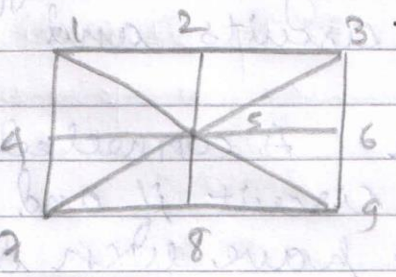
eg. find Euler path or circuit.



1-6-11-12-7-6-13-12-14-13-8-9-10-15-14-9-4-5-10-3-4-2-3-8-7-2-1

Hamilton paths.





a-b-c-d-e   1-2-3-5-9-6-5-8-7-5-4-1-3

Theorem:
Dirac's Theorem

If G is a simple Graph with 'n' vertices with $n \geq 3$ such that the degree of every vertex in G is at least $n/2$, then G has a Hamilton circuit.

Ore's theorem

If G is a simple Graph with n-vertices with $n \geq 3$ such that ~~deg(u) + deg(v)~~ $deg(u) + deg(v) \geq n$ for ~~ea~~ every pair of non-adjacent vertices u and v in G, then G has Hamilton circuit.

Planar Graph

It is a graph which can be drawn without crossing any edges.

Euler's formula:

Let G be a connected planar simples graph with 'e' edges, and 'v' vertices. Let r be the number of regions in a planar representation of G, then

$r = e - v + 2$

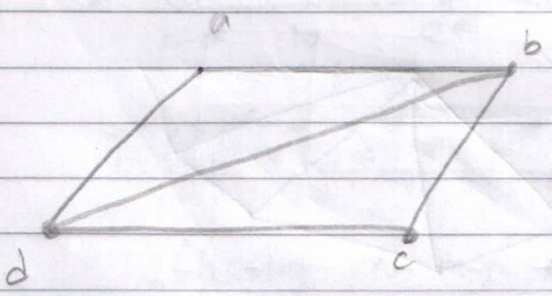
Q(1) Suppose that a connected planar graph has 30 edges. If a planar representation of these graphs divides the plane into 20 regions. How many vertices does this graph have?

Here, $e = 30$
$r = 20$
$v = ?$

By Euler's theorem,
$$r = e - v + 2$$
$$\text{or } v = e - r + 2$$
$$= 30 - 20 + 2$$
$$= 12$$

Hence, there are 12 vertices.

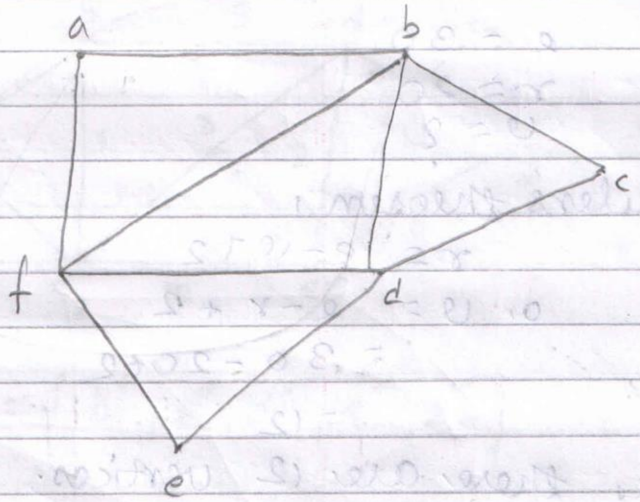
Corollary 1:-
If G is a connected planar graph with e edges and v vertices where $v \geq 3$, then $e \leq 3v - 6$



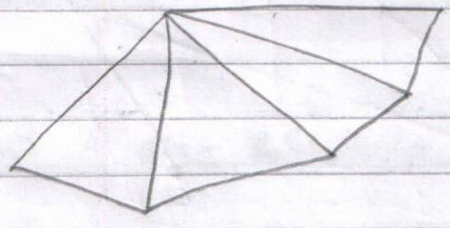
$e = 5$
$v = 4$

$$\therefore e \leq 3v - 6$$
$$5 \leq 12 - 6$$
$$5 \leq 6$$

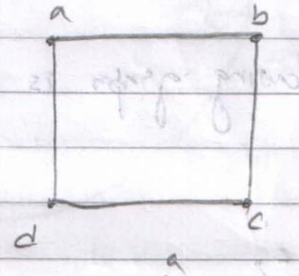
Corollary 2:-
If G is a connected planar graph, then G has a vertex of degree not exceeding 5.



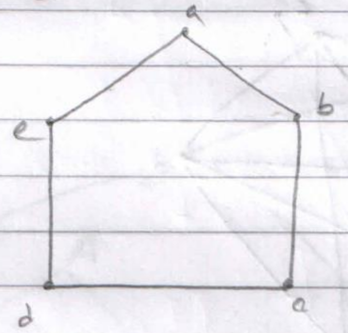
deg (a) = 2
deg (b) = 4
deg (c) = 2
deg (d) = 4
deg (e) = 2
deg (f) = 3



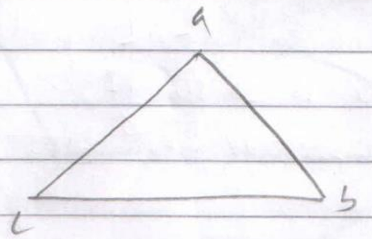
Corollary 3.
If a connected planar simple graph has e edges and v vertices with $v \geq 3$ and no circuits of length 3 then, $e \leq 2v - 4$.



$4 \leq 2 \times 4 - 4$
$4 \leq 8 - 4$
$4 \leq 4$ True

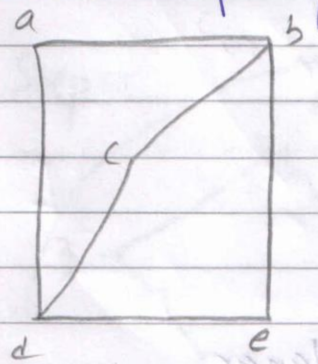


$5 \leq 2 \times 5 - 4$
$5 \leq 10 - 4$
$5 \leq 6$. True.

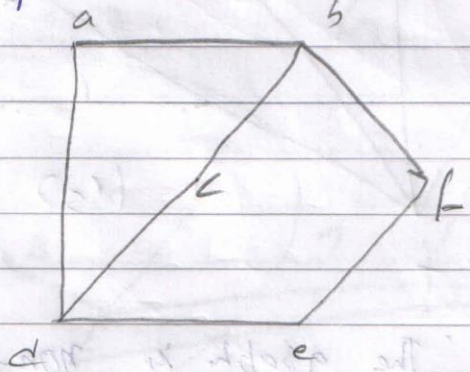


$3 \leq 2 \times 3 - 4$
$3 \leq 6 - 4$
$3 \leq 2$ which is false

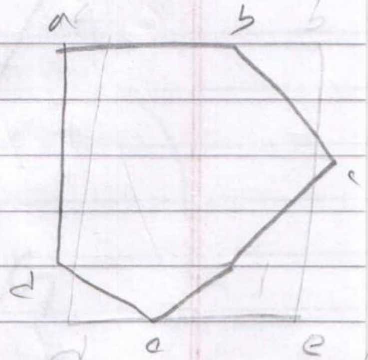
Homeomorphic graph:-



G_1



G_2



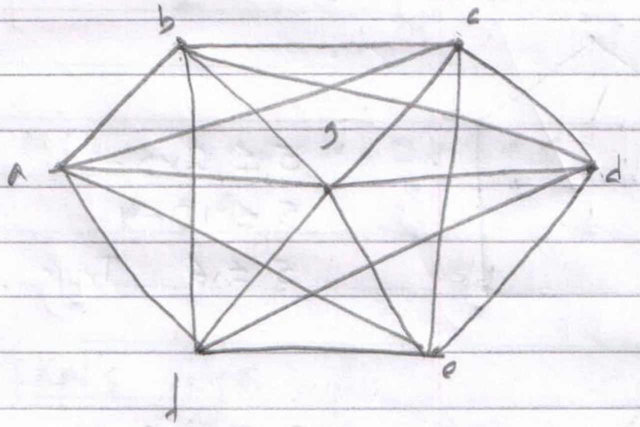
G_3

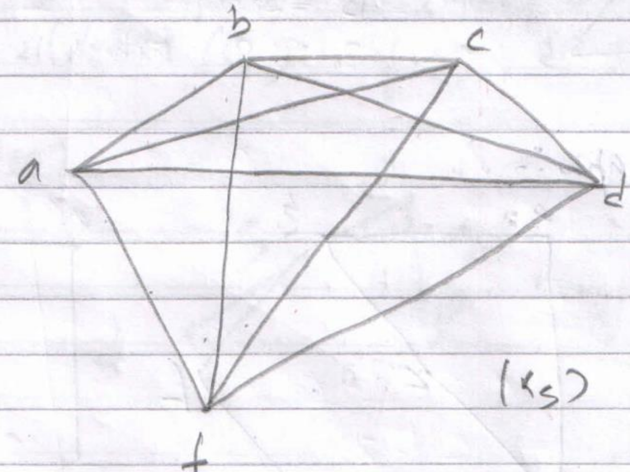
G_1 and G_2 are homeomorphic to G_3

Kuratowski's theorem:- (Planarity testing algorithm).

A graph is a non-planner if and only if it contains subgraph homeomorphic to $K_{3,3}$ or K_5.

Determine whether the following graphs is planar or not?





(K_5)

∴ The graph is non-planar.

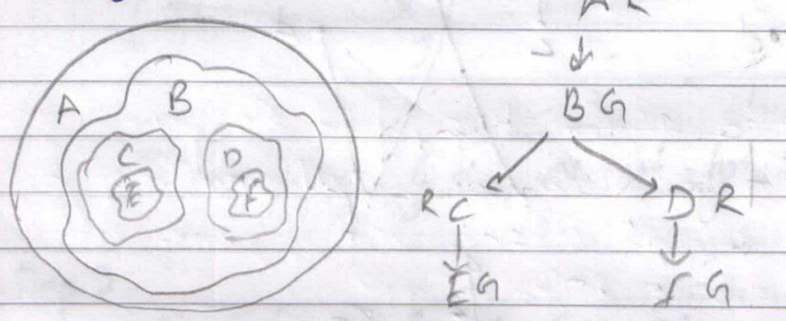
classmate
Date
Page

Graph Coloring

~~Chromatic number~~
The four color Theorem:

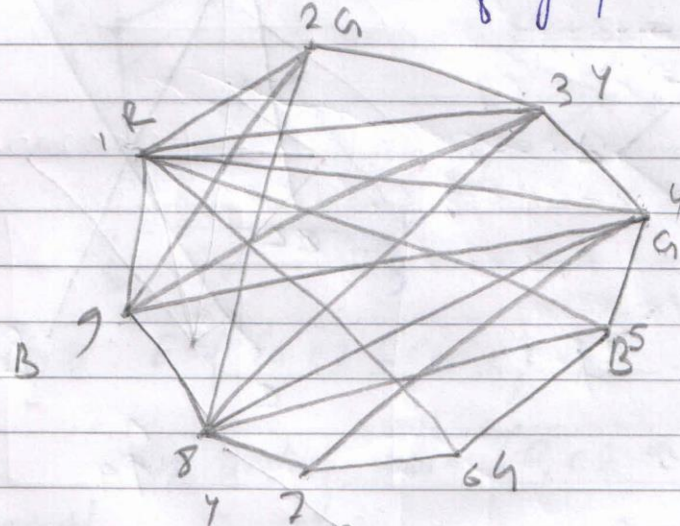
The chromatic number number of a planar graph is no greater than four.

E.g:-



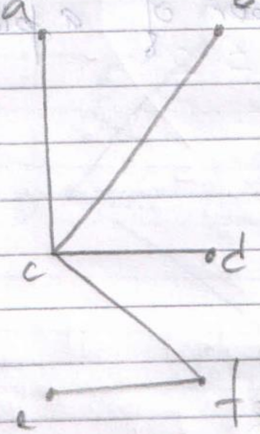
Chromatic number = 2

Find the chromatic no. of graph below:-

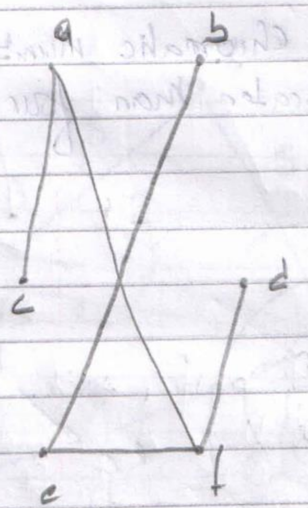


Trees

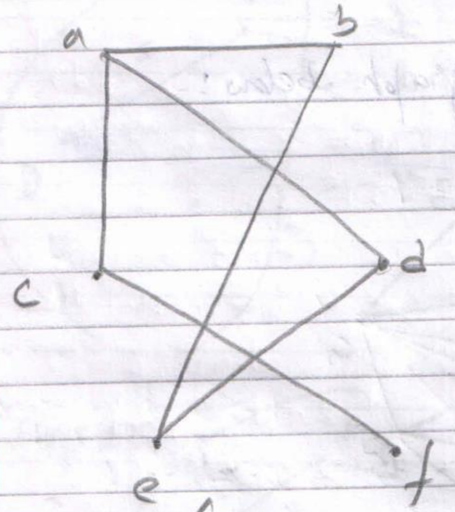
which of the graph show below are trees?



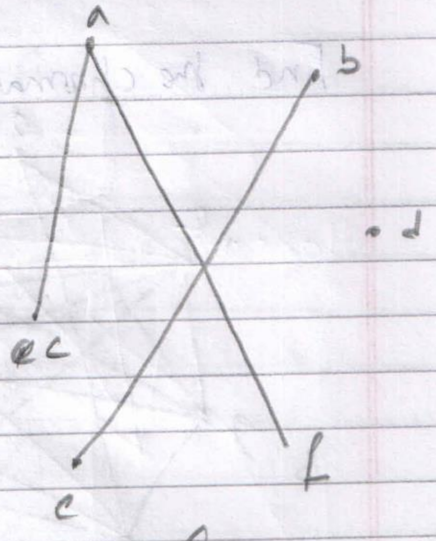
G_1 (Tree)



G_2 (Tree)

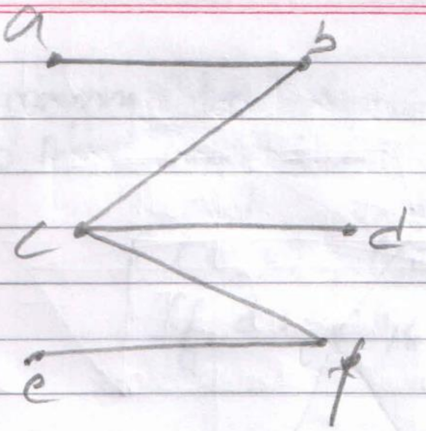


G_3 (Graph)

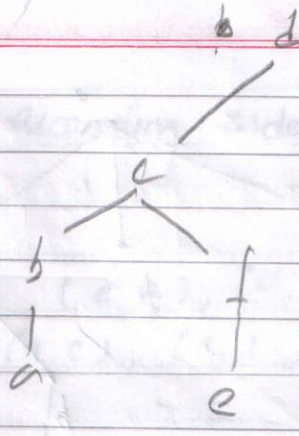


G_4 (Forrest)

Unrooted

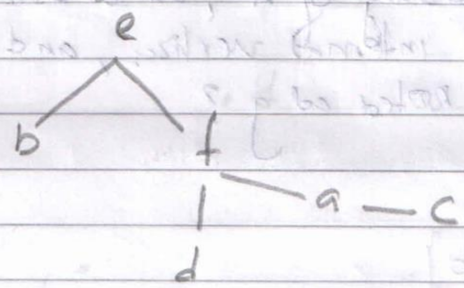


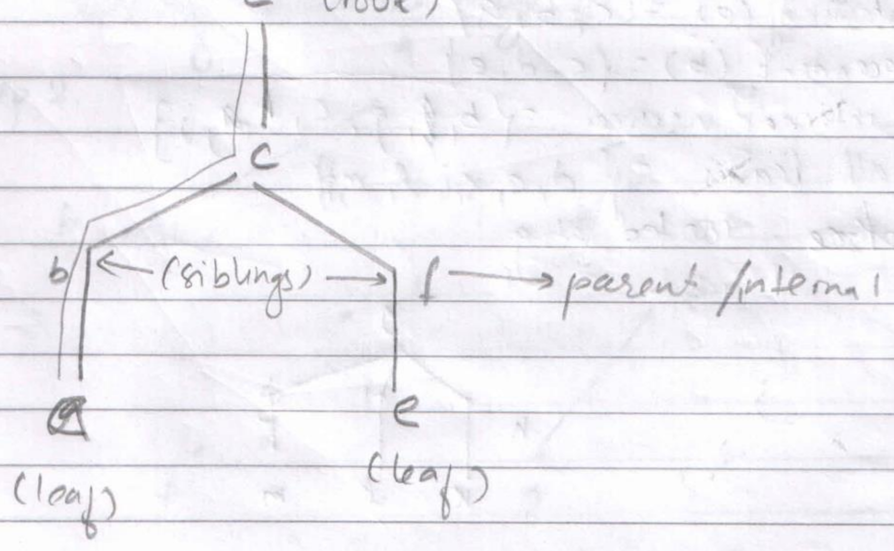
Rooted



~~Rooted graph.~~

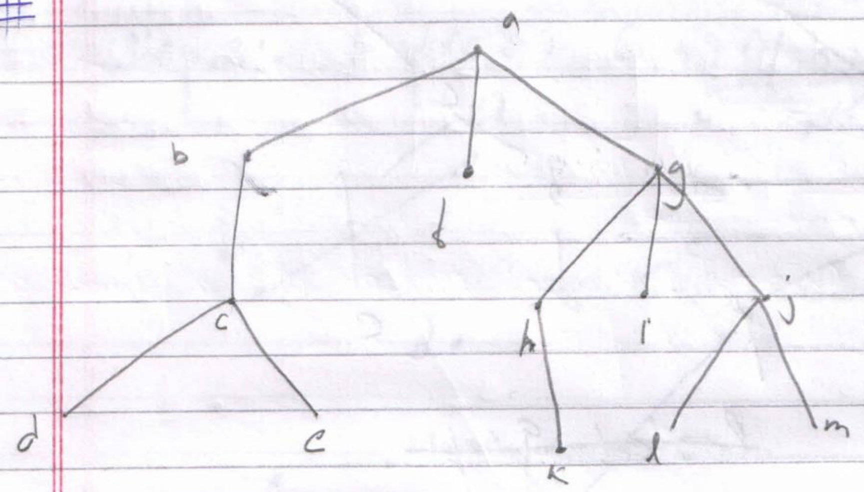
Rooted graph (G, e) of e





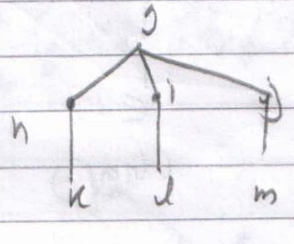
ancestor of (a) = {b, c, d}
decendencts of (d) = {c, b, a, f, e}

#



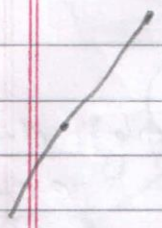
In this rooted ~~graph~~ tree, find the parent of c, children of g, the sibling of h, all ancestor of e all decendent of b, all internal vertices, and all leaves. What is the subtree rooted at g?

parent of (c) → {b}
children of (g) → {h, i, j}
sibling of (h) → {i, j}
ancestor of (e) = {c, b, a}
decendent (b) = {c, d, e}
all internal vertices = {b, f, g, c, h, a, j}
∴ all leaves = {d, e, k, i, l, m, f}
subtree rooted at g:

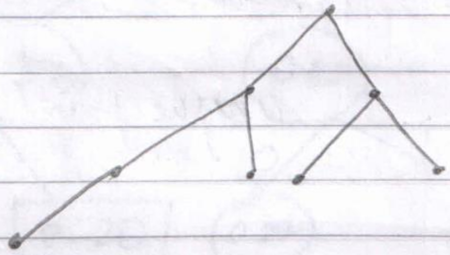


m-ary tree

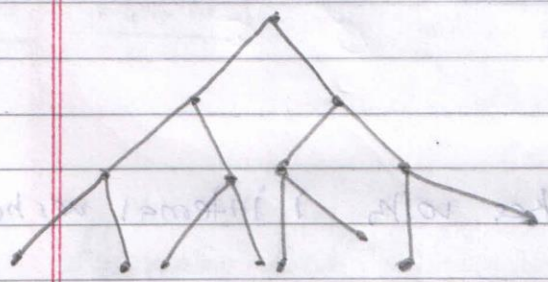
1-ary



2-ary ⇒ binary

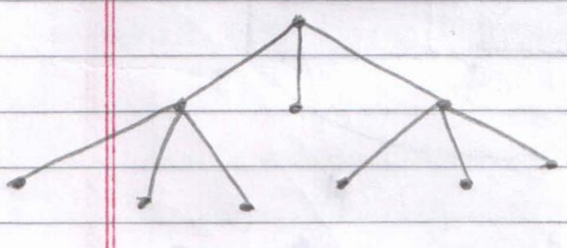


full m-ary tree (2-ary)

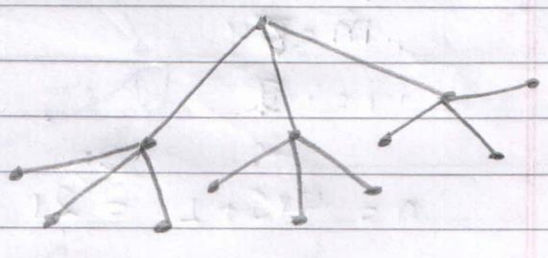


full-binary tree

3-ary tree

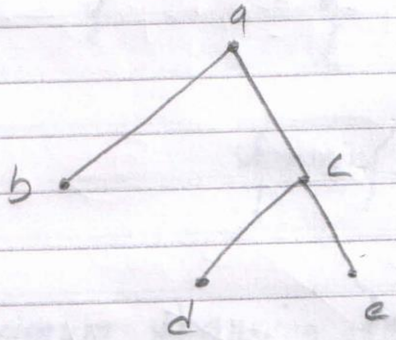


full 3-ary tree



Theorem :-

An undirected graph is a tree if and only if there is a unique simple path between any two of its vertices.

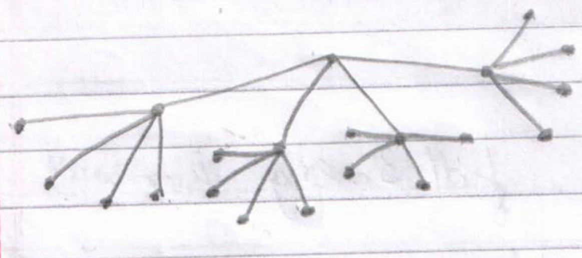


unique path (simple) : d-c-e
-e-c-a-b
= d-c-a-b

Properties of Trees :-

Theorem :-

A full m-ary tree with i internal vertices contain $n = mi + 1$ vertices

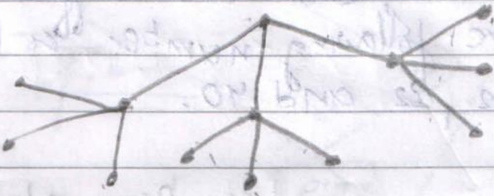


$m = \cancel{5}4$

$i = \cancel{5}$

$\therefore n = 4 \times 5 + 1 = 21$

Theorem :- A tree with n vertices has n-1 edges.



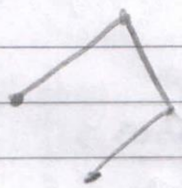
vertices (n) = 13
edges = 12 => (n-1)

Basic step: n = 1
edge = 0

Inductive hypothesis: K vertices = K-1 edges

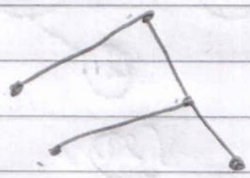
Inductive ~~step~~ step: K+1 vertices = K ~~verts~~ edges

T



K vertices
K-1 edges

T'



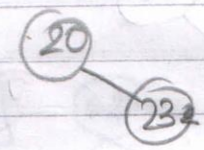
K+1 vertices
K edges

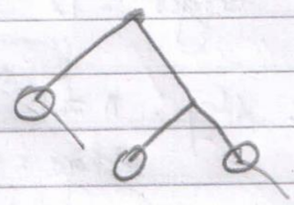
Applicatim of Trees:-
Binary Search Tree (BST)

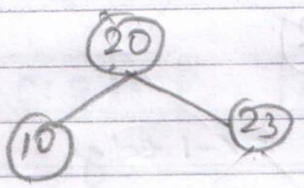
Starting with empty BST, show the effect of successively adding the following number as keys:
20, 23, 10, 21, 30, 15, 2, 22 and 40.

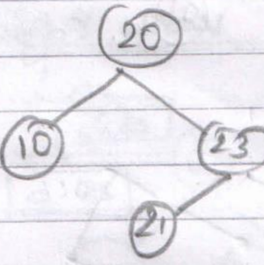
less than ⇒ left child
more than ⇒ right child

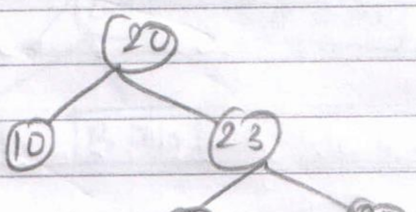
(I) 

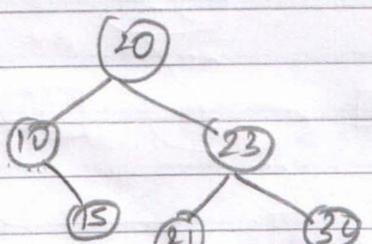
(II) 



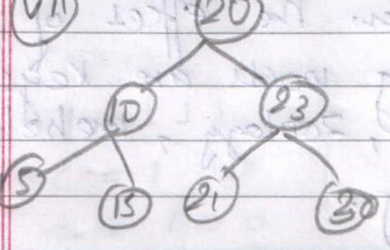
(III) 

(IV) 

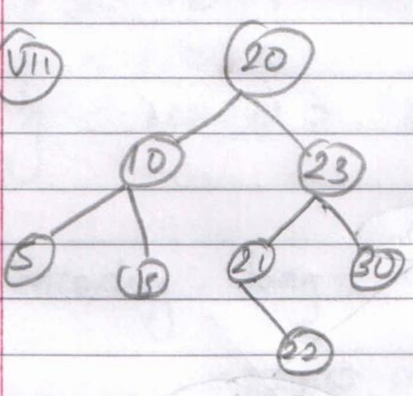
(V) 

(VI) 

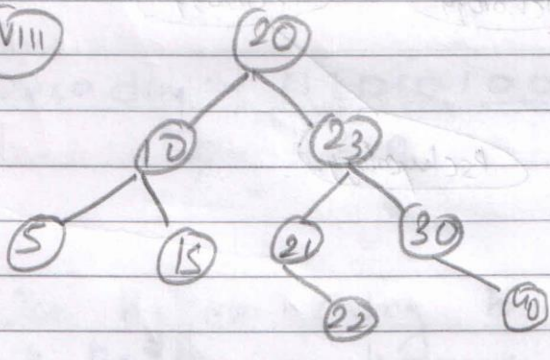
VII



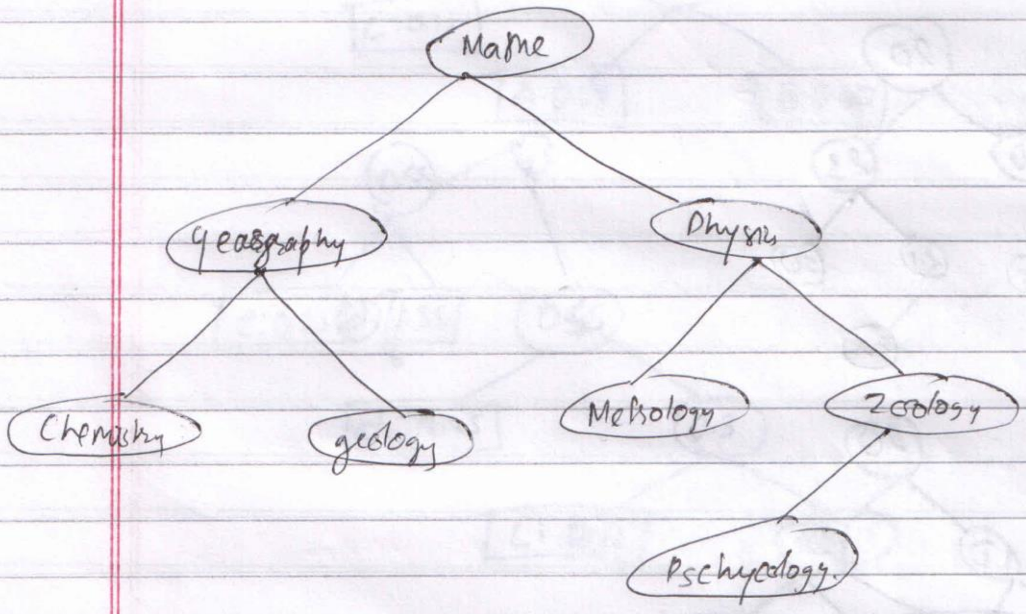
VII



VIII

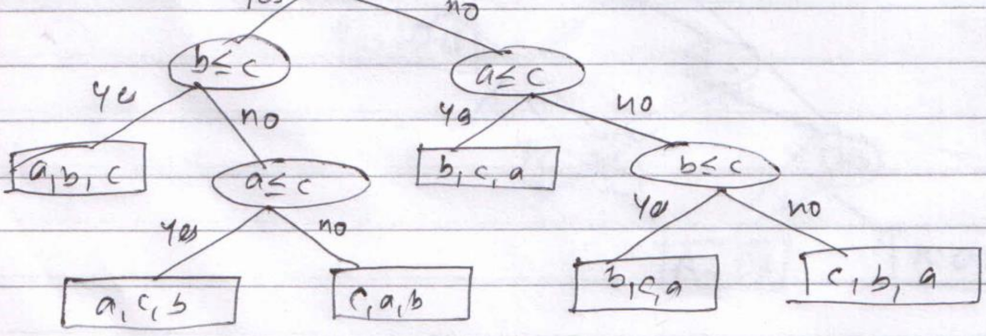


starting with empty BST, show the effect of successively adding the following words are keys :- mathematics, physics, geography, zoology, meteology, geology, psychology & chemistry.



Decision Tree.

Decision tree of sorting 3 nos.



Prefix codes.

Huffman code (tree) (Greedy paradigm / algorithm)

Eg: C:{a, b, c, d, e, f, g}

f(c): {40, 20, 15, 12, 8, 3, 2}

Construct the Huffman tree and list out prefix cods for characte.

Step 1:

$\boxed{a:40}$ $\boxed{b:20}$ $\boxed{c:15}$ $\boxed{d:12}$

$\boxed{e:8}$ $\boxed{f:3}$ $\boxed{g:2}$

Step 2:

Arranging the characters with the frequencies in ascending order.

- Prefix codes:-

  a = 0
  b = 111
  c = 110
  d = 100
  e = 1011
  f = 10101
  g = 10100

Encoding: bag

= 111 0 10100

Decoding:- 111010100
⇒ b a g

Use Huffman coding to encode the following symbols with the ~~frequency~~ frequencies listed.

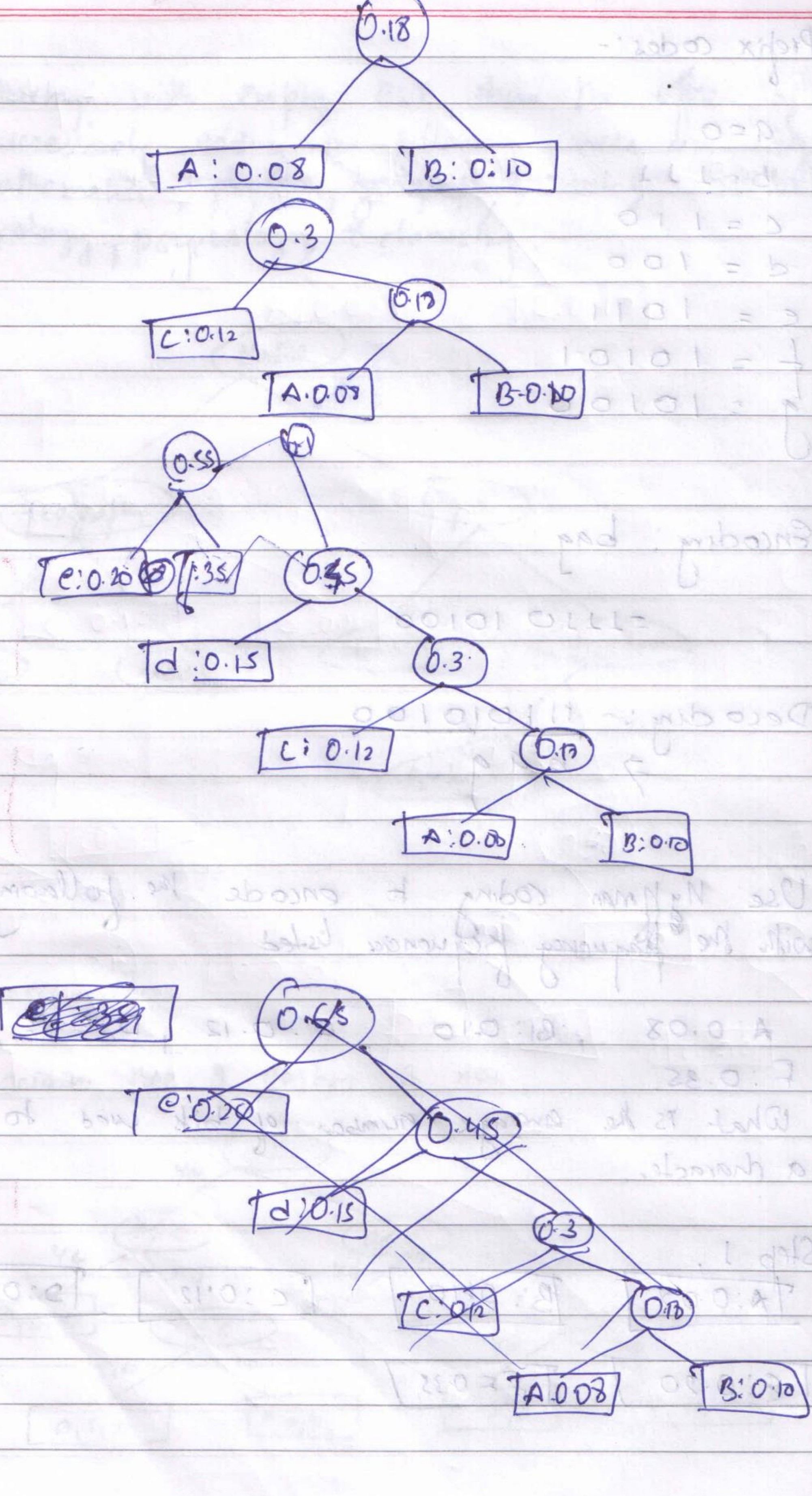
A: 0.08, B: 0.10, C: 0.12, D: 0.15, E: 0.20, F: 0.35.

What is the average number of bits used to encode a character.

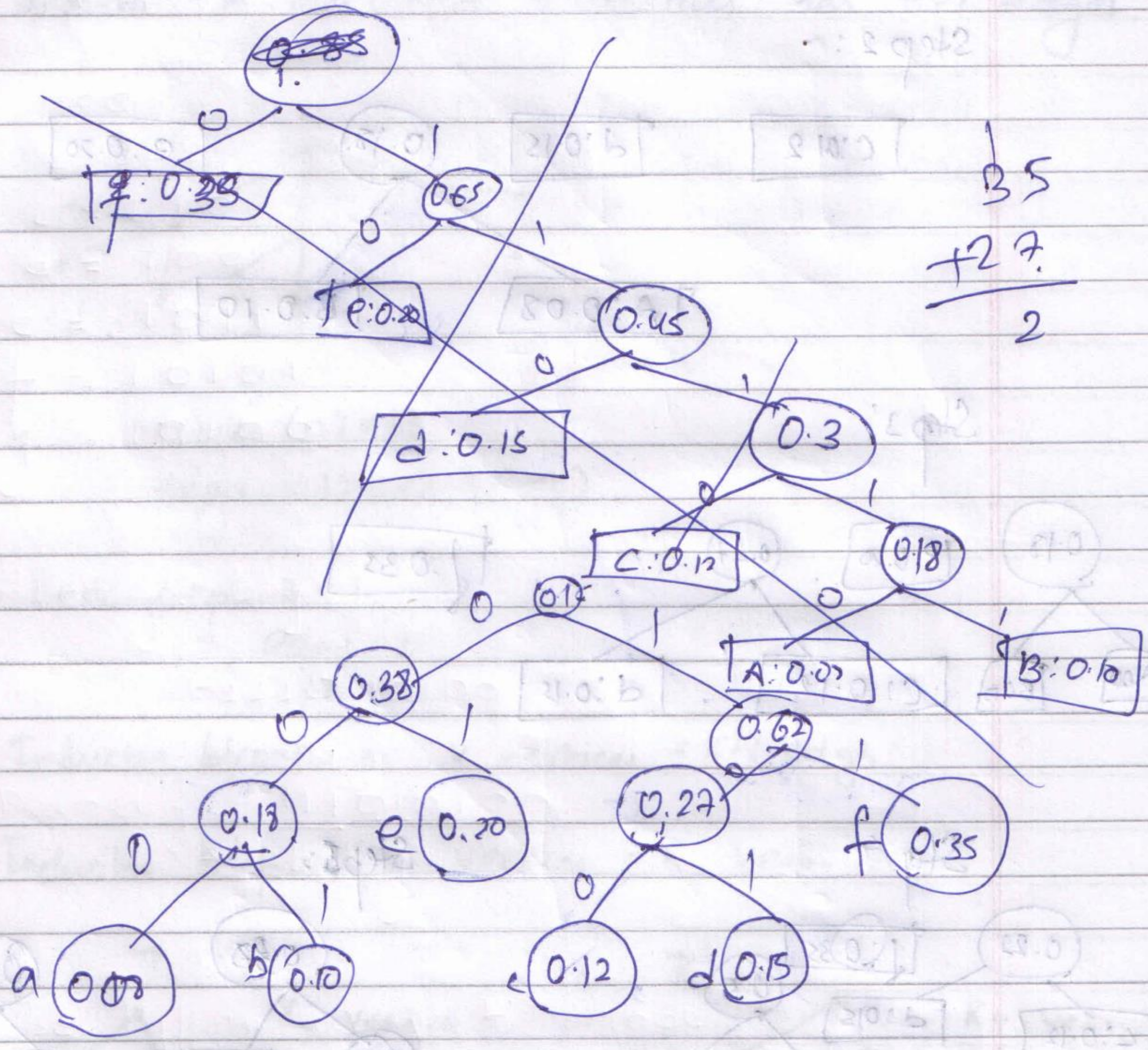
Step 1.

A: 0.08 | B: 0.10 | C: 0.12 | D: 0.15

E: 0.20 | F: 0.35


0.18
A : 0.08
B : 0.10
0.3
0.18
C : 0.12
A.0.08
B.0.10
0.55
e : 0.20
0.45
d : 0.15
0.3
C : 0.12
0.18
A : 0.08
B : 0.10
0.65
e : 0.20
0.45
d : 0.15
0.3
C : 0.12
0.18
A.0.08
B : 0.10

$$\frac{35 + 27}{2}$$



Prefix code:

$a = 000$

$b = 001$

$c = 100$

$d = 101$

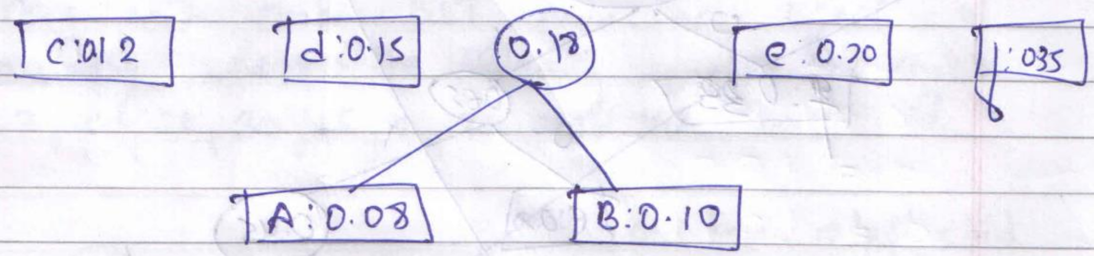
$e = 01$

$f = 11$

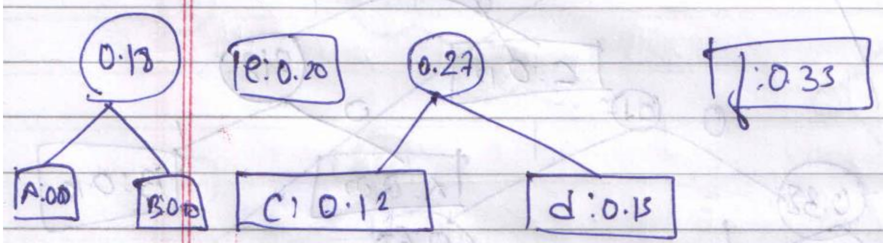
$\therefore$ The no. of bits $= 3+3+3+3+2+2$

$$\text{Average} = \frac{3+3+3+3+2+2}{6} = \frac{\cancel{16}\,8}{\cancel{6}\,3} = \frac{8}{3}$$

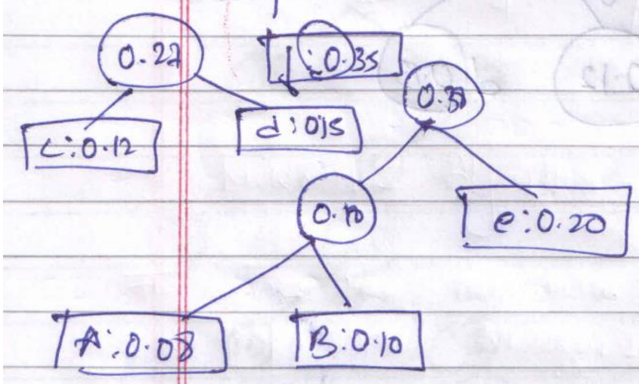
Step 2:



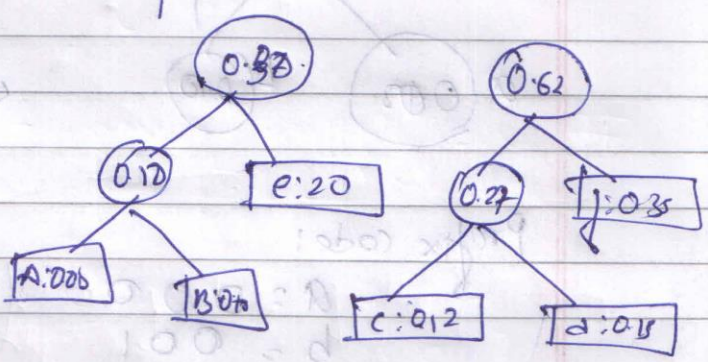
Step 3:



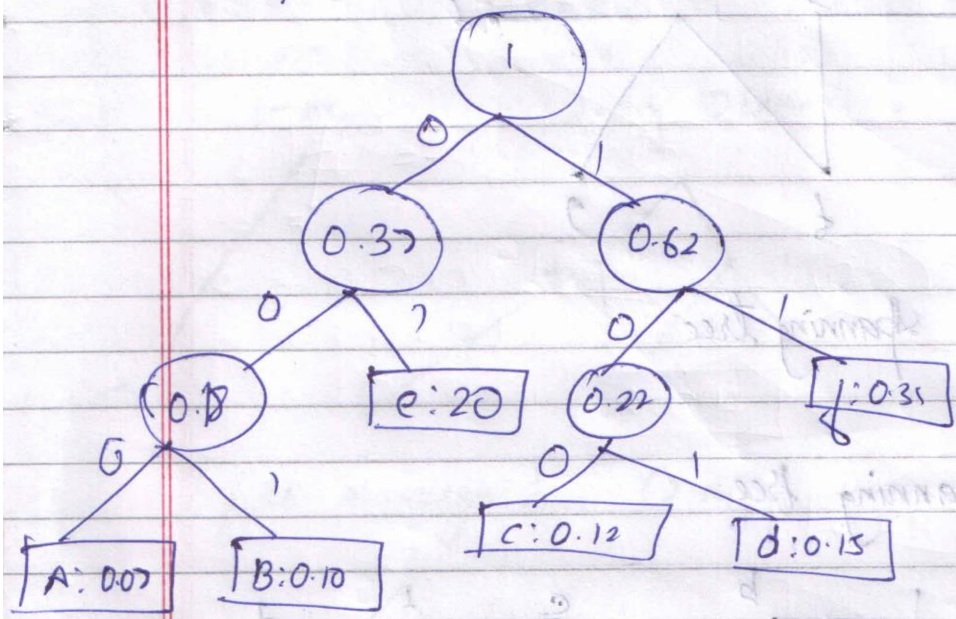
Step 4:



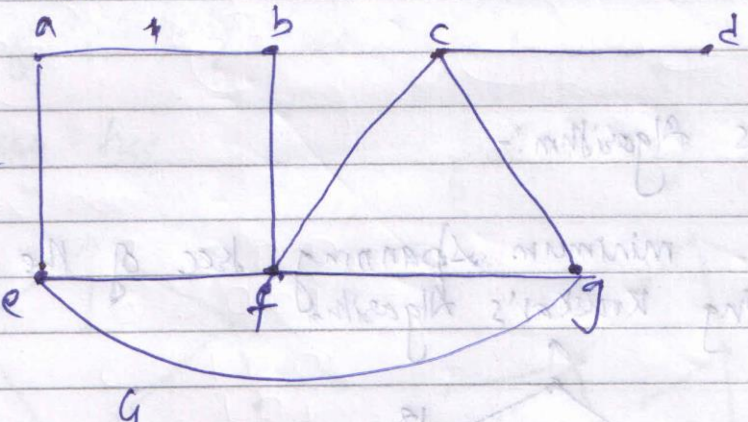
Step 5:



Step 6:

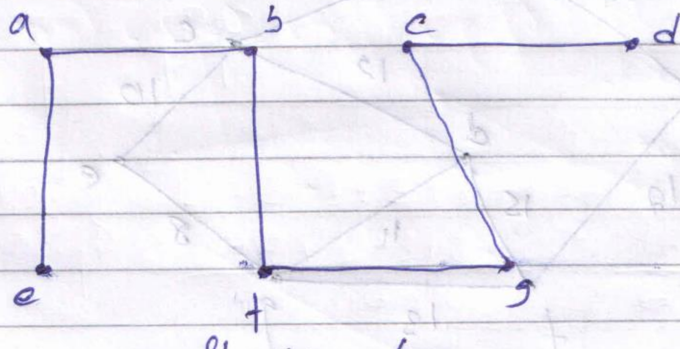


Spanning tree :-

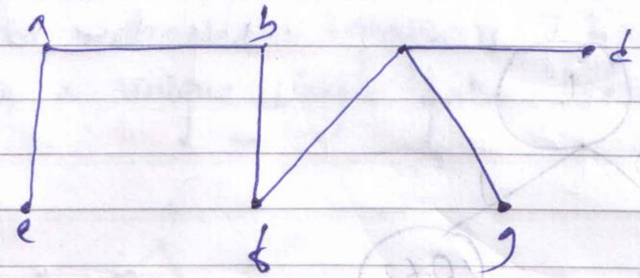


G

T ⇒

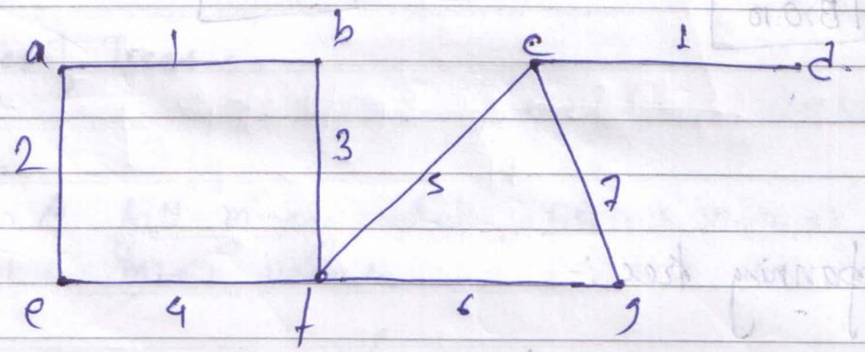


Spanning tree.



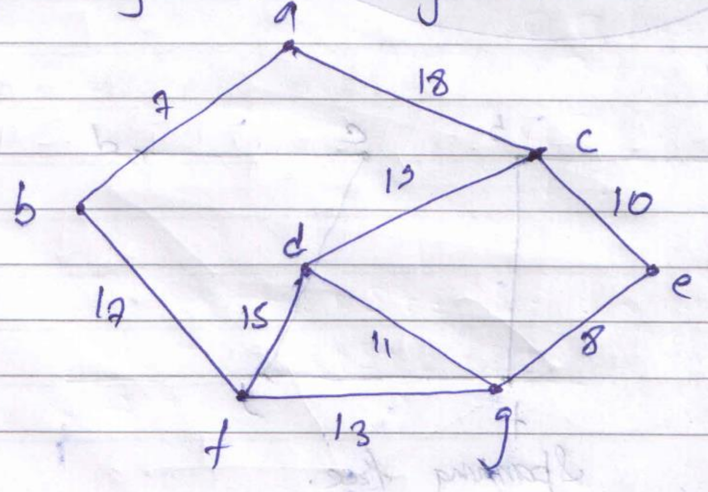
Spanning tree

Minimum spanning tree:



Kruskal's Algorithm:-

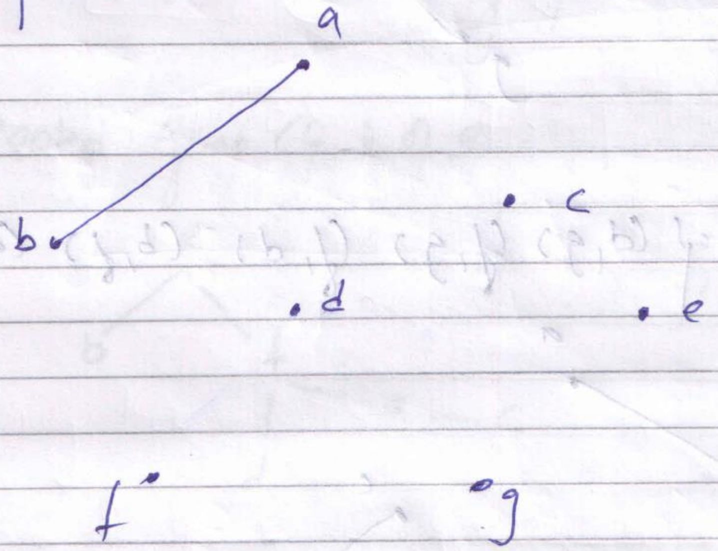
Find the minimum spanning tree of the following graph using Kruskal's Algorithm



Arranging the edges in ascending order with respect to their weight.

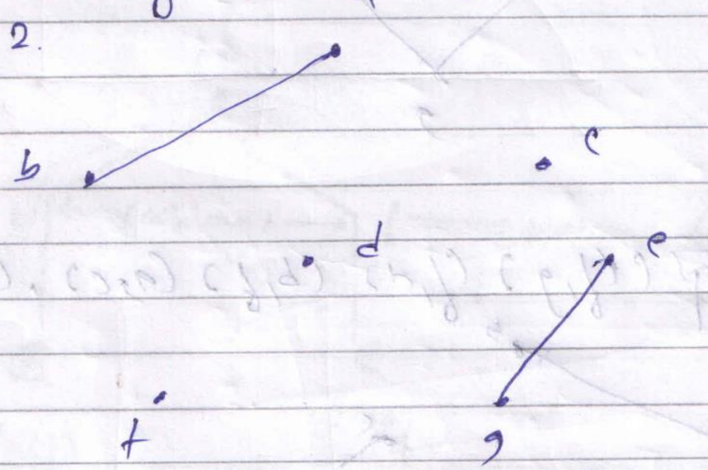
$\{(a,b), (e,g), (c,e), (d,g)\ (f,g), (f,d), (b,f), (a,c), (c,d)\}$

Step 1:



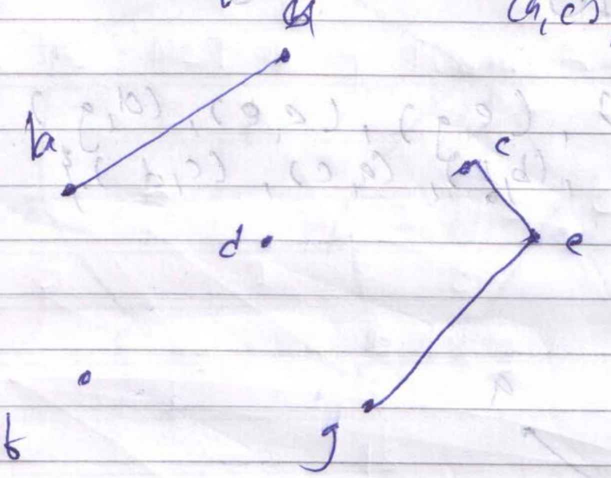
Selecting minimum edge from the set, E.

Step 2.

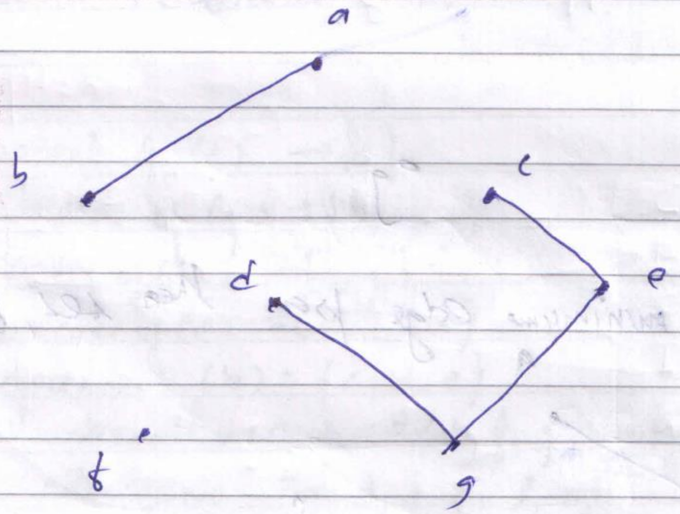


$E = \{(e,g), (c,e), (d,g), (f,g)\ (f,d)\ (b,f), (a,c)\ (c,d)\}$

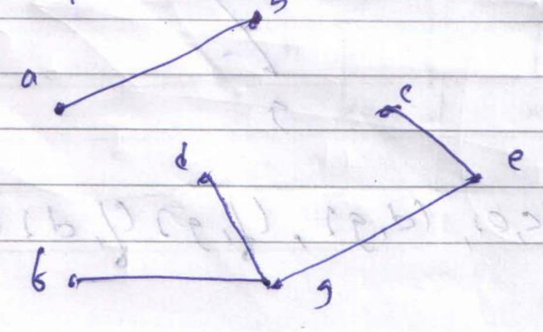
Step 3:- $E = \{(c,e), (d,g) (f,g), (f,d) (b,f), (a,c), (e,d)\}$



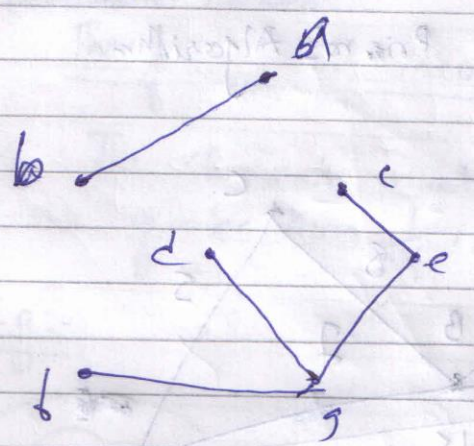
Step 4: $E = \{(d,g) (f,g) (f,d) (b,f) (a,c), (e,d)\}$



Step 5: $E = \{(f,g) (f,d) (b,f) (a,c), (e,d)\}$



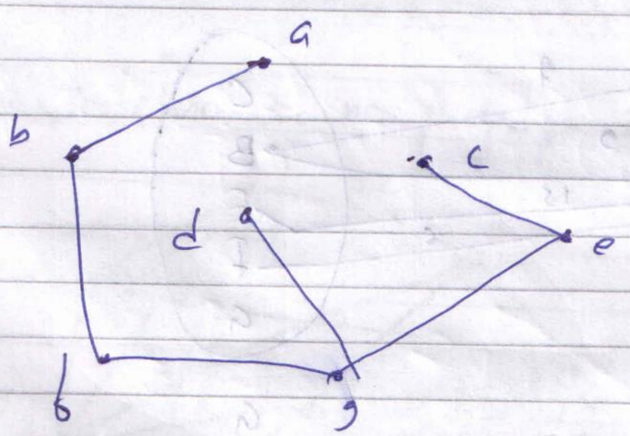
Step 6: E = { (f, d), (b, f) (a, c) (c, d) }



Discarding (f, d) as it forms a circuit.

Step 7:

E = { (b, f) (a, c) (c, d) }



Step 8: E = { (a, c) (c, d) }

Discarding both (a, c) & (c, d) as it forms circuit.

∴

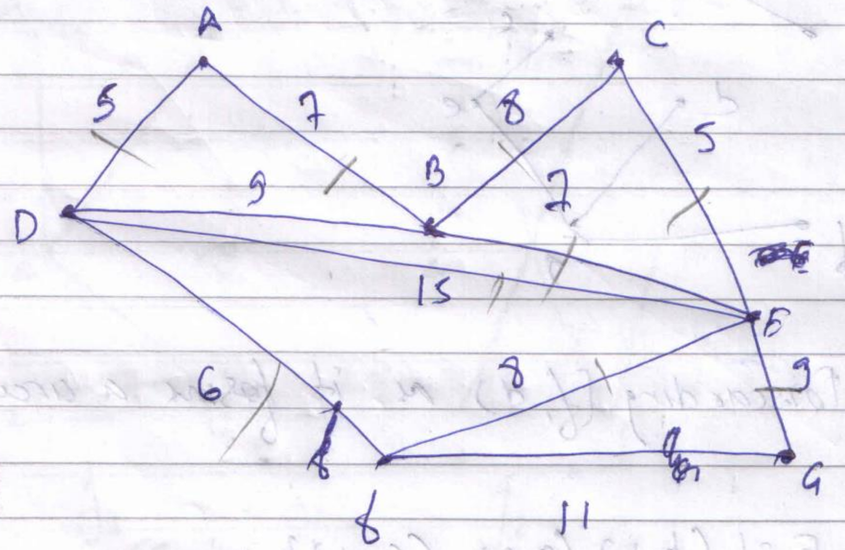
$MST = 7 + 17 + 13 + 8 + 10 + 11$

$= 66$,,

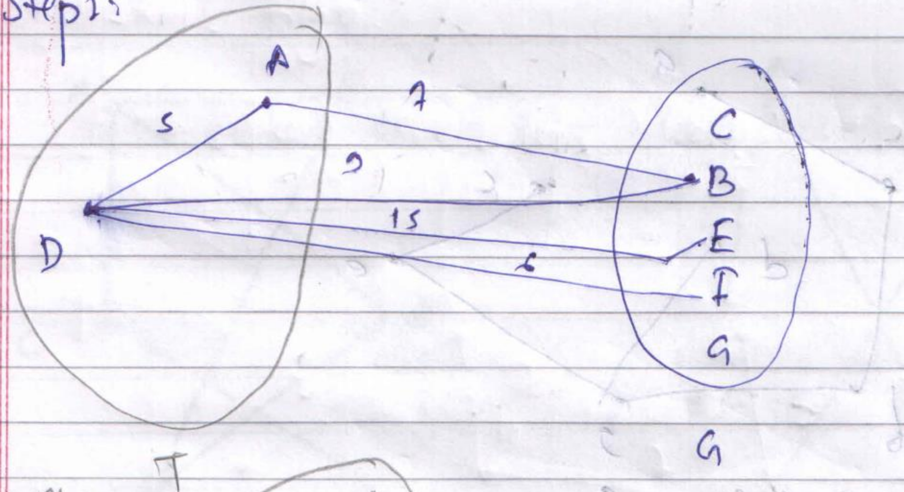
Graph Cutting Algorithm

Priam's Algorithm. (~~cutting graph~~)

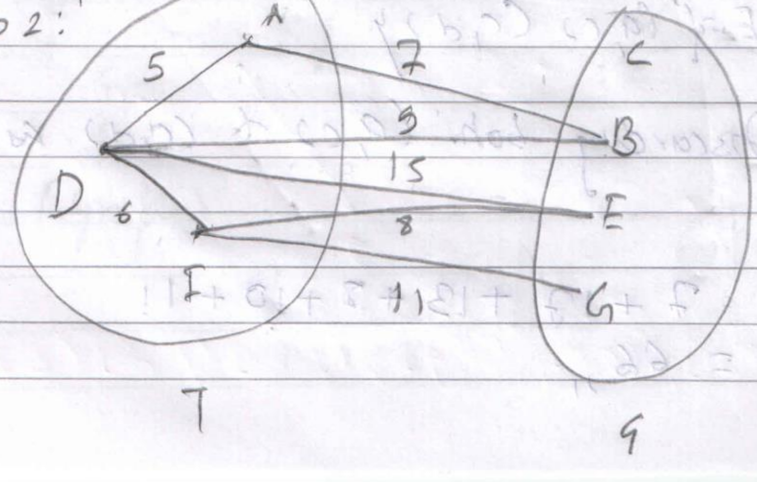
Find the MST using Priam's Algorithm



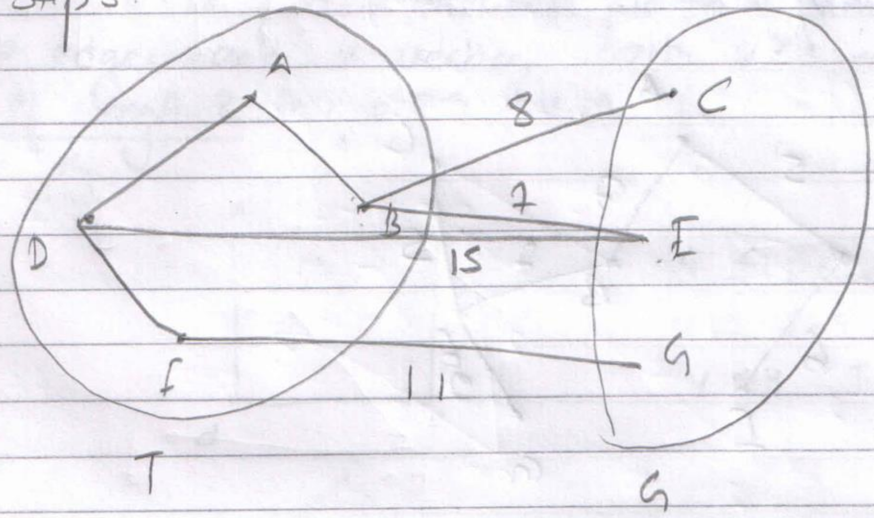
Step 1:



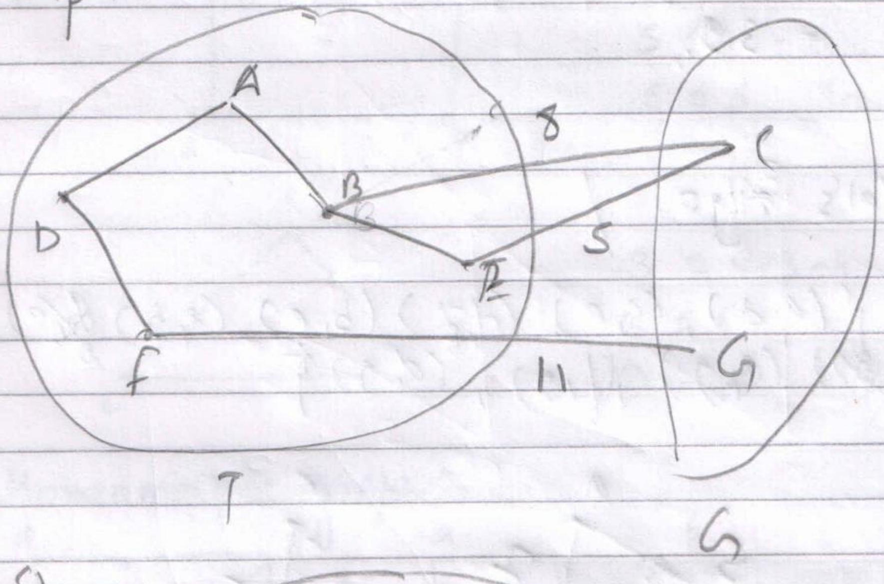
Step 2:



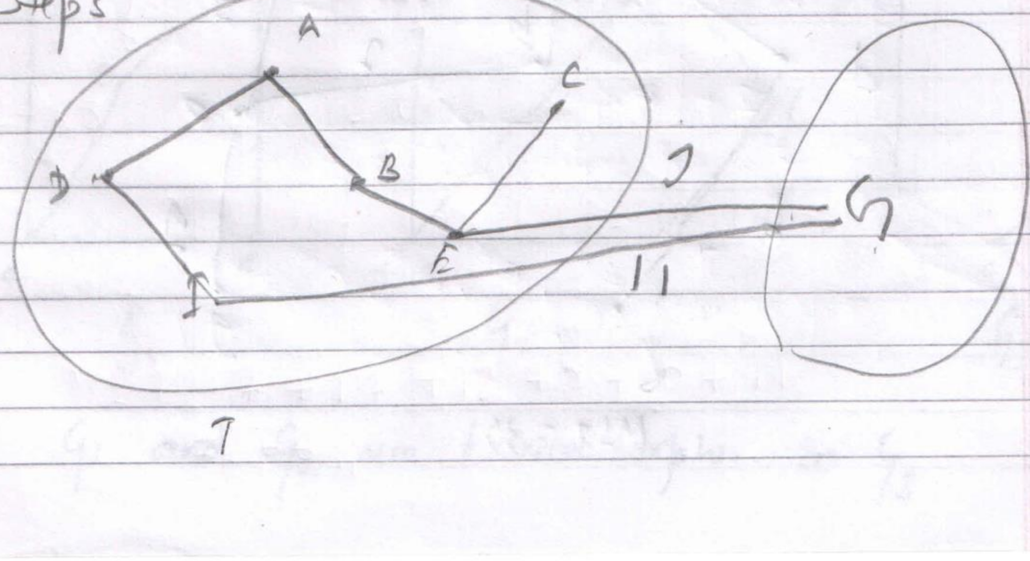
Step 3.



Step 4:

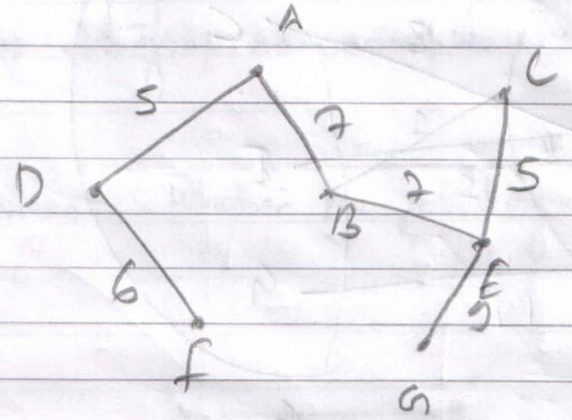


Step 5



Step 6:

∴ The



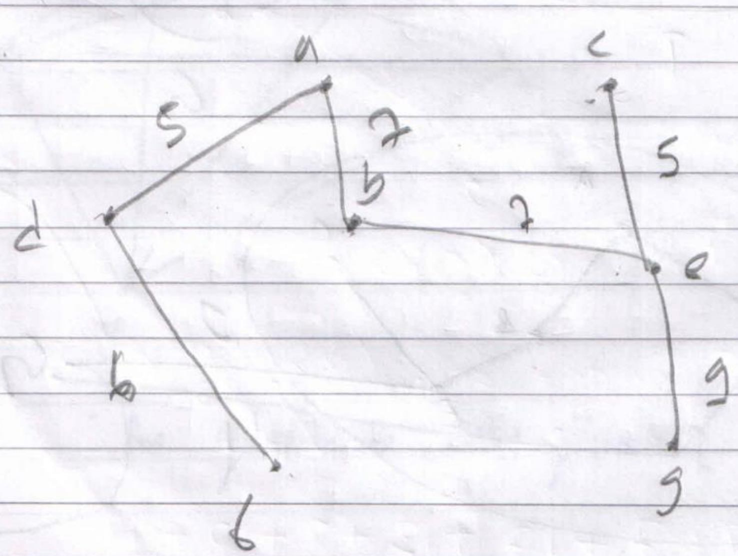
$MST = 5+6+7+7+9+5$

$= 39$,,

Kurskal's Algo.

$E = \{(a,d), (c,e), (d,f), (b,e), (a,b), \cancel{f(e)}, \cancel{(b,e)}, \cancel{(d,b)}, (e,g), (f,g), (d,g)\}$

Step 7.

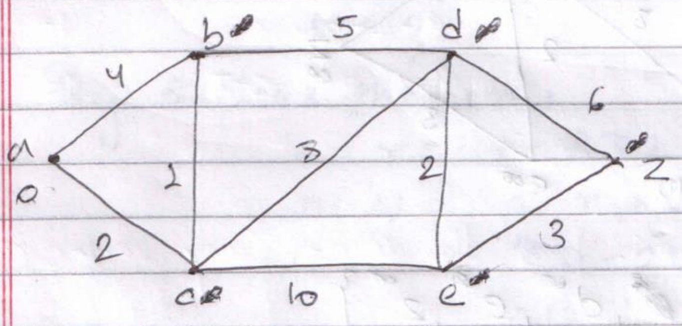


MST = 39

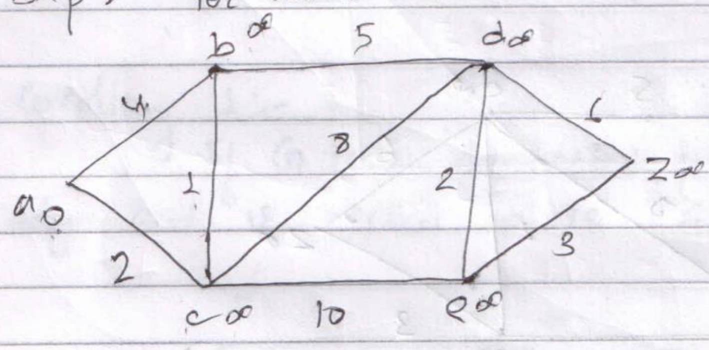
~~Step~~ :

Shortest part Algorithm :-

Dijiksha's Algorithm.



Step 1: ~~For value~~ vertex a:-



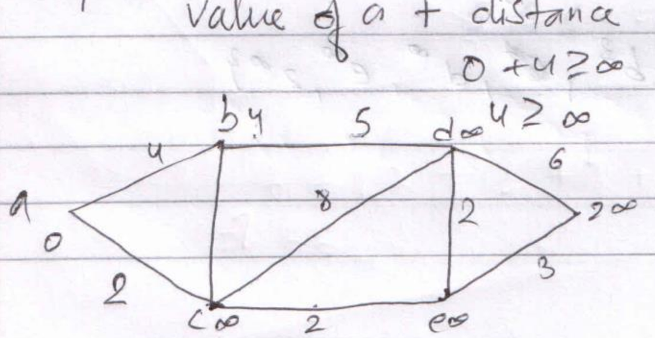
$T = \{a^0, b^\infty, c^\infty, d^\infty, e^\infty, z^\infty\}$

Choosing minimum distance value.

Step 2:-

value of a + distance of edge $\geq \infty$

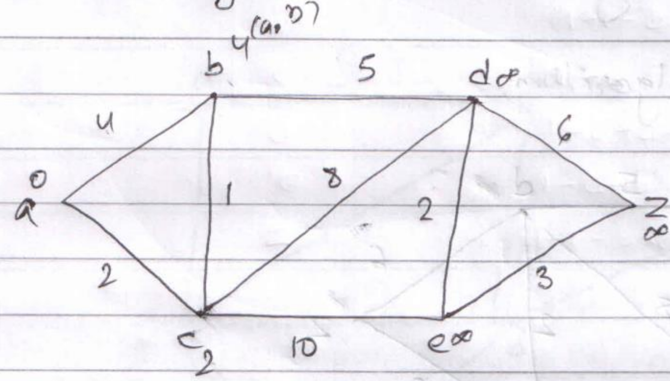
$0 + 4 \geq \infty$



$T = \{b^4, c^\infty, d^\infty, e^\infty, z^\infty\}$

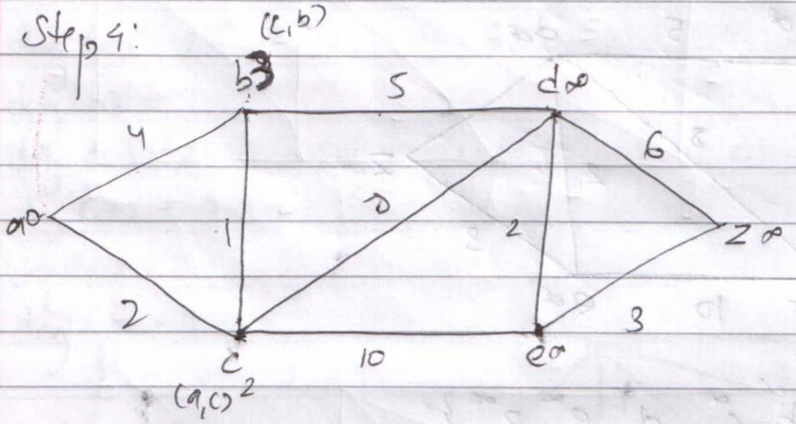
$T = \{ b^4, c^\infty, d^\infty, e^\infty, z^\infty \}$

Step 3: Choosing min distance value.



$T = \{ b^4, \cancel{c}, d^\infty, e^\infty, z^\infty \}$

Step 4:



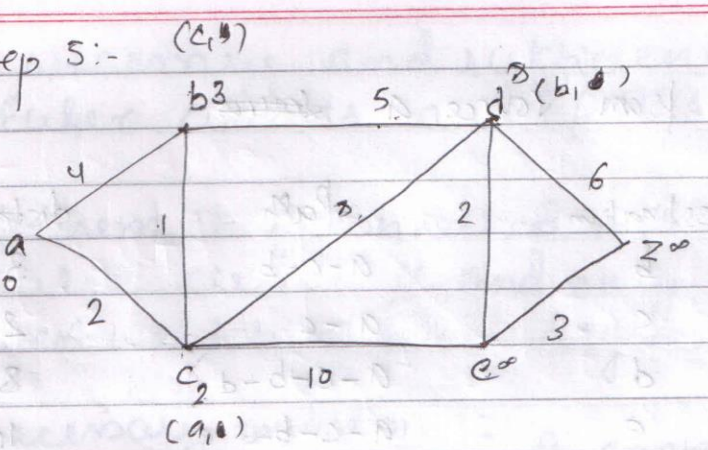
distance of c + wt. of edge < distance of b.

$2 + 1 < 4$

∴ distance of b is 3

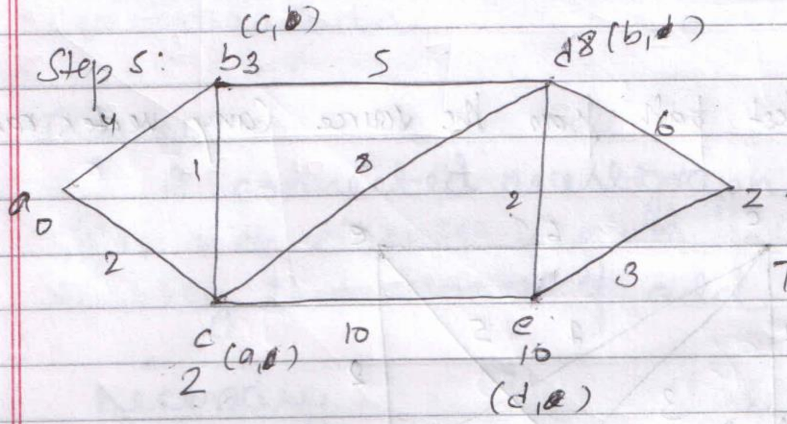
$T = \{ b^3, \cancel{c}, d^\infty, e^\infty, z^\infty \}$

Step 5:-



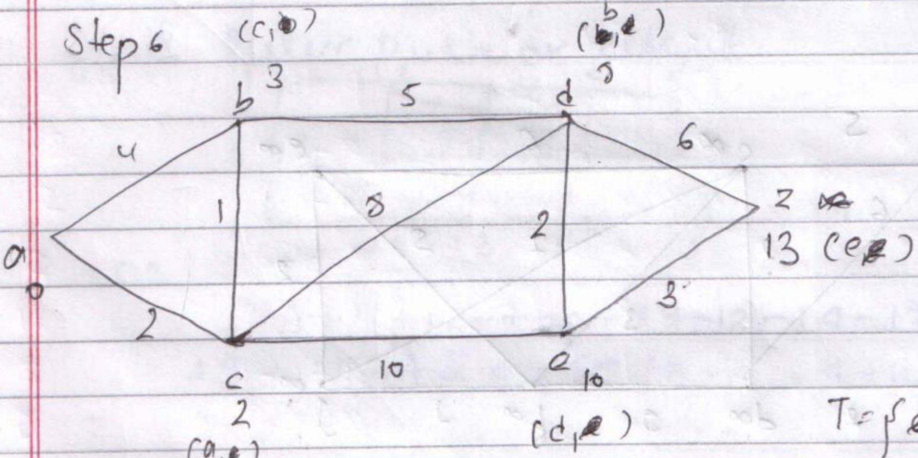
$T=\{\cancel{b}, \cancel{c}, d^{8}, e^{\infty}, z^{\infty}\}$

Step 5:



$T=\{e^{10}, z^{\infty}\}$

Step 6

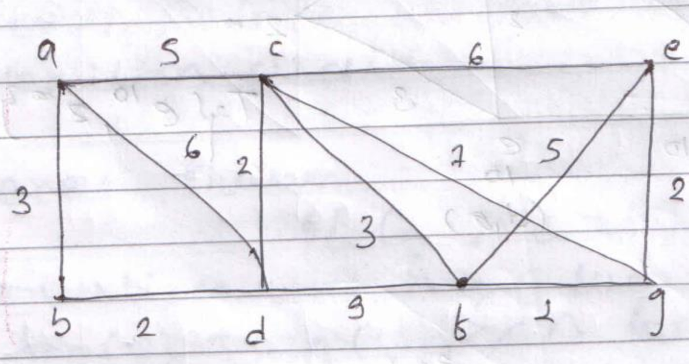


$T=\{\cancel{e}\ \cancel{z}^{13}\}$

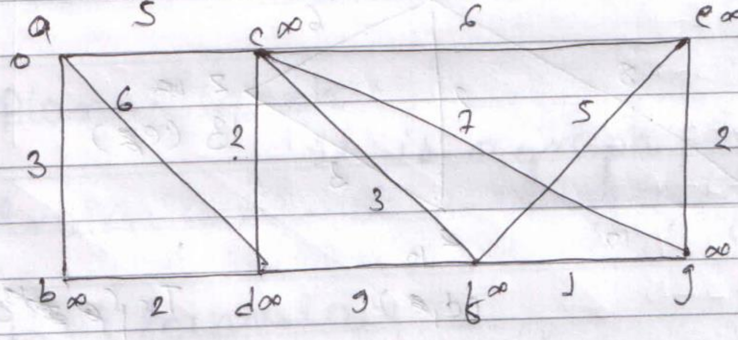
Shortest path from source a ~~source~~.

Source	Destination	Path	distance
a	b	a-c-b	3
a	c	a-c	2
a	d	a-c-b-d	8
a	e	a-c-b-d-e	10
a	z	a-c-b-d-e-z	13

Find the shortest path from the source (any vertex randomly)

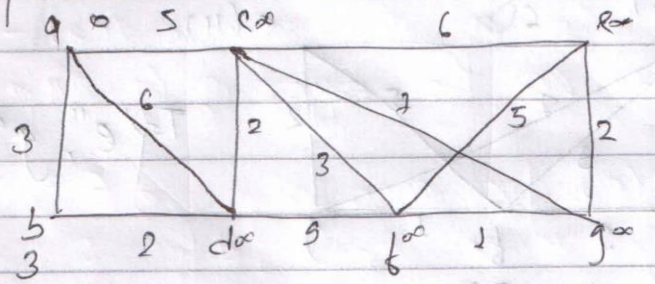


Step 1.



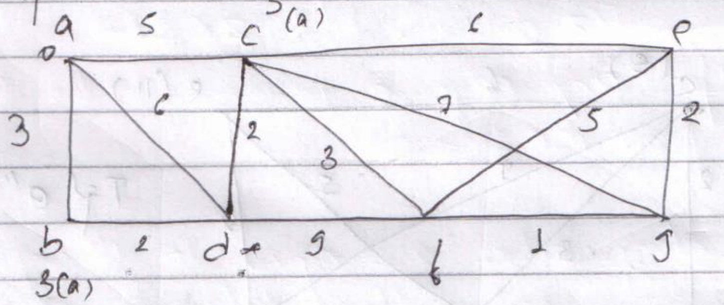
$T = \{a^0, b^\infty, c^\infty, d^\infty, e^\infty, f^\infty, g^\infty\}$

Step 2:



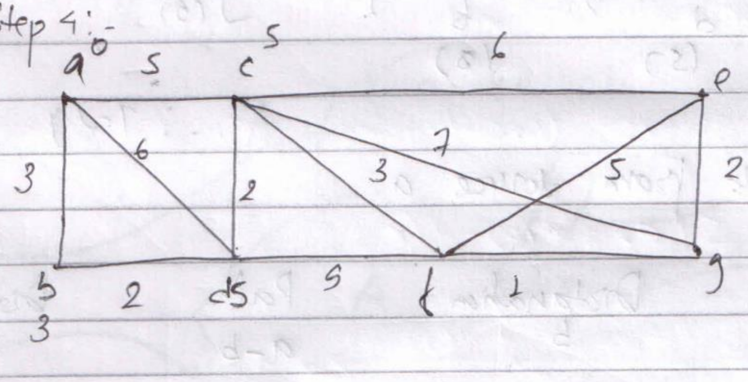
$T = \{b^3, c^\infty, d^\infty, e^\infty, f^\infty, g^\infty\}$

Step 3:



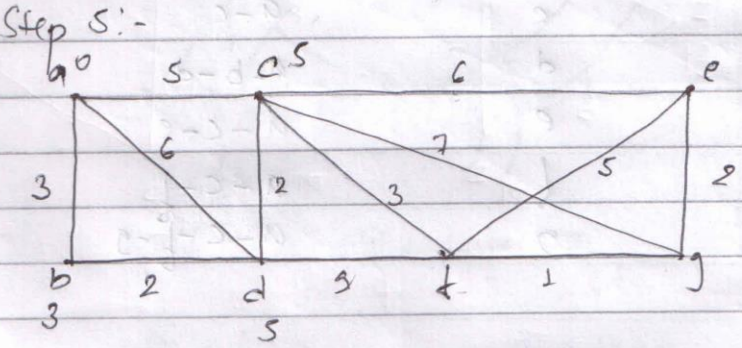
$T = \{c^5, d^\infty, e^\infty, f^\infty, g^\infty\}$

Step 4:-



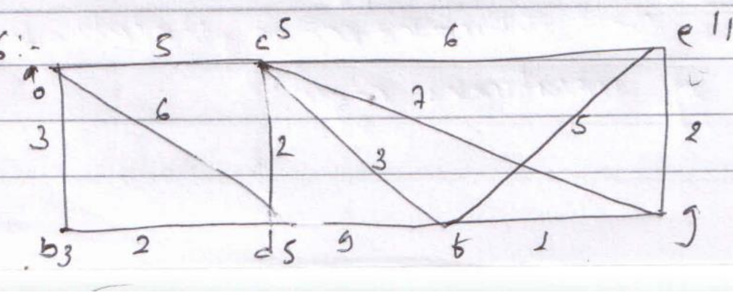
$T = \{c^5, \cancel{d^\infty}\, d^5, e^\infty, f^\infty, g^\infty\}$

Step 5:-



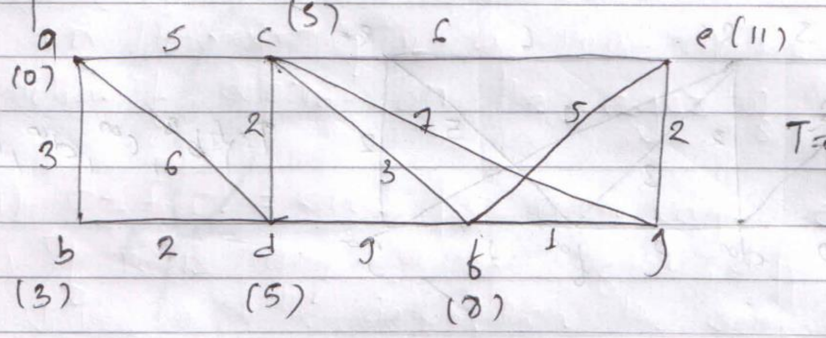
$T = \{c^5, e^\infty, f^\infty, g^\infty\}$

Step 6:-



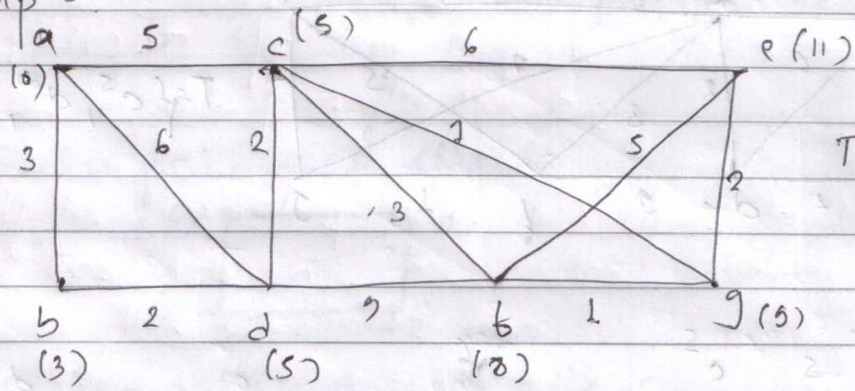
$T = \{e^{11}, f^\infty, g^\infty\}$

Step 7:



$T = \{e^{11}, g^{\infty}\}$

Step 8:



$T = \{e^{11}\}$

$\therefore T = \{\}$

Shortest path from source a

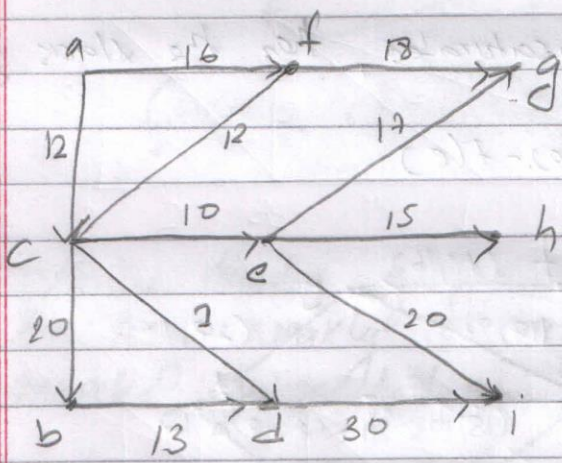
Source	Distanation	Path	Distance
a	b	a-b	3
a	c	a-c	5
a	d	a-b-d	5
a	e	a-c-e	11
a	f	a-c-f	8
a	g	a-c-f-g	9

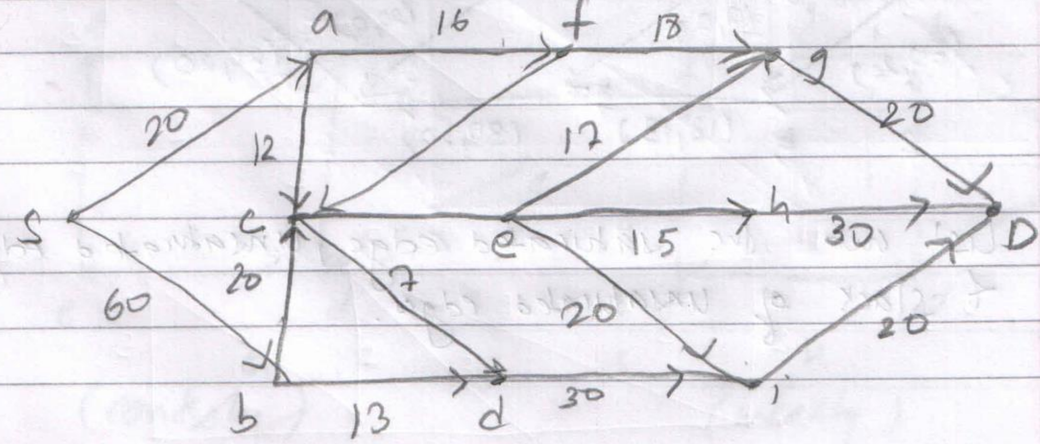
Network flow graph

A directed graph 'G' that is weakly connected and contains no loops. is called network ~~flow~~ iff

1. there are two distinguished vertices S & D of G, called source and sink of G respectively.
2. There is a non-negative real valued function k defined on the edge of G.

→ the function k is called the capacity function of G and if e is any ~~element~~ edge of G, the value k(e) is called the capacity of e.
→ the vertices distinct from source S & the sink D are called intermediate vertices.





Flow :-

A flow in G is a non-negative real-valued function f defined on the edges of G such that

(i) $0 \le f(e) \le k(e)$ for each edge $e \in E(G)$.

(ii) If x is any vertex of G, different from the source or the sink, then the sum all values $f(x,y)$ such that $y \in A(x)$ must equal the sum of all values ~~F(x,y)~~ $f(z,x)$ such that $z \in B(A)$

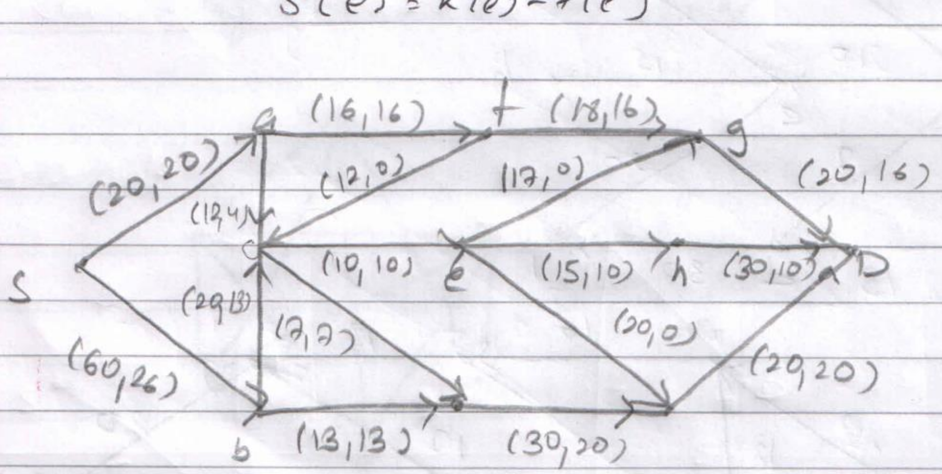
iii) $F(e) = 0$ for any edge e incident to the source S or incident from the sink D.

→ If the flow along edges are equal to its capacity, then such edges are called saturated edges & if not then unsaturated edges.

→ If an edge is unsaturated, then the slack of e in a flow F is

$$S(e) = K(e) - F(e)$$

Eg:



List out the saturated edge, unsaturated edges & slack of unsaturated edges.

Solution:-

saturated edges:- (s→a)
(a→t)
(c→e)
(i→D)
(b→d)
(c→d)

unsaturated edges :- (s→b) (e→g)
(a→c) (e→h)
(c→b) (h→D)
(f→c) (g→D)
(d→i)
(e→i)
(f→g)

slack of (60, 26) = k(e) − f(e)
= 60 − 26
= 34

slack of (12, 4) = k(e) − f(e)
= 12 − 4
= 8

slack of (20, 13) = k(e) − f(e)
= 20 − 13
= 7

slack of $(12,0) = K(e) - f(e)$
$= 12 - 0$
$= 12$

slack of $(30,20) = K(e) - f(e)$
$= 30 - 20$
$= 10$

slack of $(18,16) = 2$

slack of $(17,0) = 17$

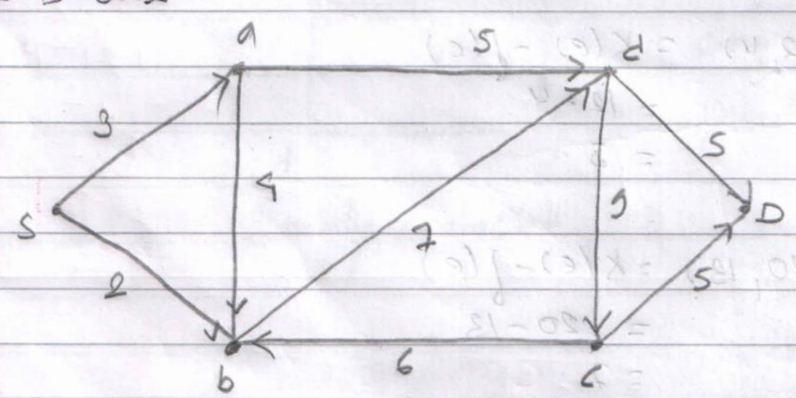
slack of $(15,10) = 5$

slack of $(20,0) = 20$

slack of $(20,16) = 4$

slack of $(30,10) = 20$

S-D cuts

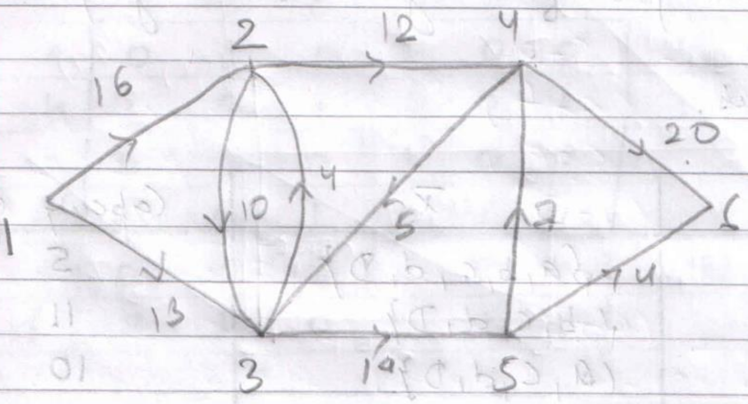


Determine the max. flow of the given network graph.

Possible S-D cuts.

X	$\bar{X}$	Capacity of $(X, \bar{X})$
{S}	{a, b, c, d, D}	5
{S, a}	{b, c, d, D}	11
{S, b}	{a, c, d, D}	10
{S, c}	{a, b, d, D}	16
{S, d}	{a, b, c, D}	19
{S, a, b}	{c, d, D}	~~14~~ ~~16~~ 12
{S, a, c}	{b, d, D}	22
{S, b, d}	{a, c, D}	~~27~~ 17
{S, b, c}	{a, d, D}	~~21~~ 15
{S, a, d}	{b, c, D}	~~25~~ ~~18~~ 20
{S, a, b, c}	{d, D}	~~23~~ ~~8~~ 17
{S, c, d}	{S, a, b, D}	~~15~~ 21
{S, b, c, d}	{a, D}	~~20~~ 13
{S, a, c, d}	{b, D}	22
{S, a, b, d}	{c, D}	14
{S, a, b, c, d}	{D}	10

min capacity is the maximal flow. = 5



X	$\bar{X}$	Capacity $(x, \bar{x})$
{1}	{2, 3, 4, 5, 6}	29
{1, 2}	{3, 4, 5, 6}	~~23~~ 35
{1, 3}	{2, 4, 5, 6}	~~34~~ ~~35~~ 34
{1, 4}	{2, 3, 5, 6}	54
{1, 5}	{2, 3, 4, 6}	40
{1, 2, 3}	{4, 5, 6}	26
{1, 2, 4}	{3, 5, 6}	~~84~~ 48
{1, 2, 5}	{3, 4, 6}	46
{1, 3, 4}	{2, 5, 6}	~~40~~ 54
{1, 3, 5}	{2, 4, 6}	31
{1, 4, 5}	{2, 3, 6}	58
{1, 2, 3, 4}	{5, 6}	34
{1, 2, 4, 5}	{3, 6}	~~33~~ ~~23~~ ~~53~~ 52
{1, 3, 4, 5}	{2, 6}	44
{1, 2, 3, 5}	{4, 6}	23
{1, 2, 3, 4, 5}	{6}	24

min ~~MAX~~ cap = 24

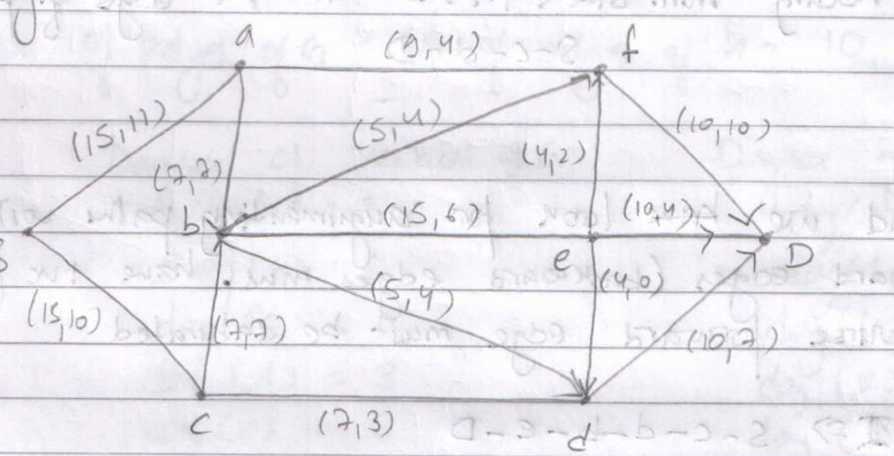
Mott (Network flow)
→ reference book.

Max-flow and min-cut theorem

This theorem asserts the following :-

1. The existence of a minimal cut $(X, \bar{X})$.
2. The existence of a maximal flow F,
3. The equality of $|F|$ and $K(X, \bar{X})$ for any maximal flow F and any minimal cut $(X, \bar{X})$

Eg:

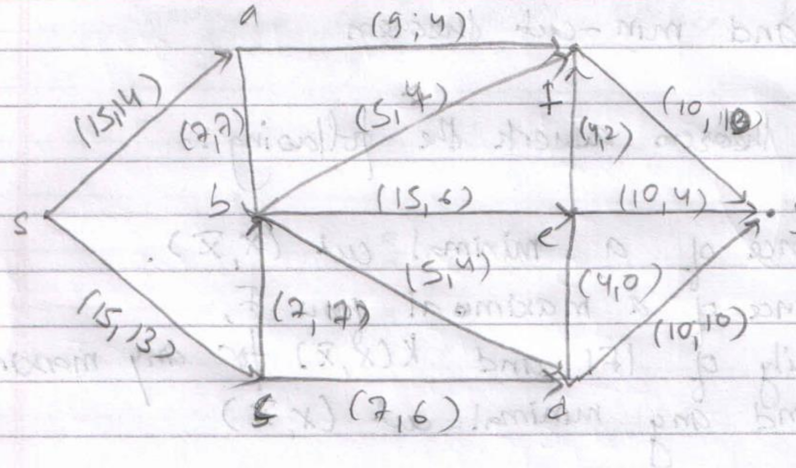


→ First of all look for an ~~aug~~ augmen~~ting~~ path where if possible all edges are forward edges & they have to be unsaturated.

Path P_1 :- S-c-d-D

Now, calculating the slack of each edge of the path P_1.

slack of (S-c) :- 5
" " (c-d) :- 4
" " (d-D) :- 3



Adding min. slack i.e. 3 at the edges of path
S-c-d-D

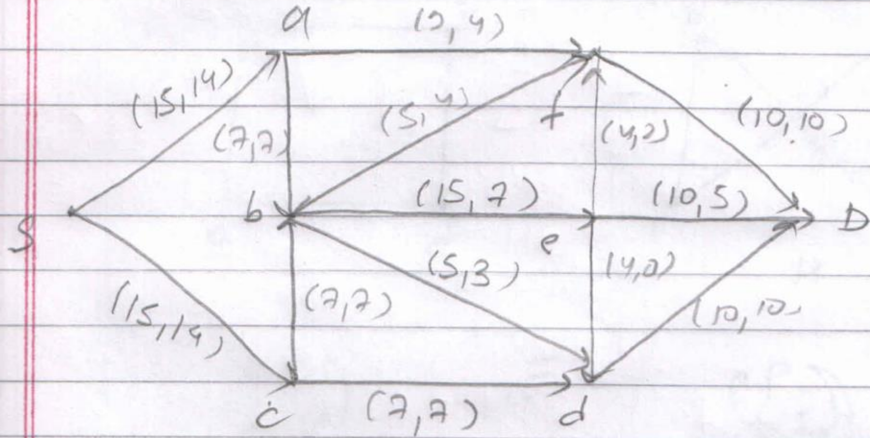
Second we have look for augumenting paths with some backward edges (backward edges must have +ve flow & of course forward edge must be saturated

Path P1 => S-c-d-b-e-D
Path P2 => S-a-f-e-D
Path P3 => S-a-f-b-e-D

Now choosing Path P1 => S-c-d-b-e-D

slack (S,c) :- 2
(c,d) :- 1
(d,b) :- 1
(b,e) :- 9
(e,D) :- 6

Now increasing the flow of edges by minimum slack and decreasing the flow of backward edges by minimum slack



Again taking path PII: S - a - f - e - D.

Slack of (S-a) = 1

" " (a-f) = 5

(f-e) = 2

(e-D) = 5

Again increasing the flow of edges by min. slack & decreasing the flow of backward edges by min. slack.