THEORY OF STRUCTURES-I

80 + 20 + 25

Theory, Ass. (14+3+3: Ass, Assign, Attendence), Practical

Chapters:

1	Introduction.	4hr	5 marks
2	Strain Energy Method	4 hr	5 marks
3	Virtual work Method	6 hr	10 marks
4	Deflection of Beam	7-8 hr	15 marks
5	Influence line diagram	10 hr	20 marks
6	Three hinged arch.	7 hr	15 marks
7	Suspension Cable System	7 hr	10 marks

Books:

1. S.S. Bhavikatti
2. Suresh Hada (Nepali)
3. C.S. Reddy
4. Ramamrutham
5. Norrish
6. Darkov
7. B.C. Punmia
8. A.K. Jain

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Structures:

Structure is the combination (composition) of various elements like beam, frame, truss, arch, cable, etc. which is capable of transferring the applied loads to the foundation without exceeding the permissible limit of deformation and vibration.
Eg. Building, Bridge, Dams, Towers, canals, etc.

Structural analysis:

Structural analysis is the process of calculating the internal stress parameters of various structural components like axial force, shear force, bending moment, etc. In structural analysis; we also calculate the deformation due to such stresses. It can be done by manual methods or through computer programs. Various softwares ~~Types of structures:~~ like SAP, STAAD, ETABS, SAFE are developed for structural analysis. For this analysis; we need to input the material property (E), Geometric property (Dimensions), Boundary conditions (Supports), loads and the stress parameters and deformation are obtained as output.

Real structures are 3-D structures which has high number of unknowns. So; manual analysis is impracticable and uneconomic. Hence; we go for the analysis through computers.

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Types of structures:

① Based on materials used:

ⓐ Masonry structures. (Brick, mud related)
ⓑ. Timber structures.
ⓒ R.C.C. structures.
ⓓ. Fibre glass structures.
ⓔ. Steel structures.

②. Based on indeterminancy:

ⓐ. Determinate structures.
ⓑ Indeterminate structures

Methods / Approaches of structural analysis:

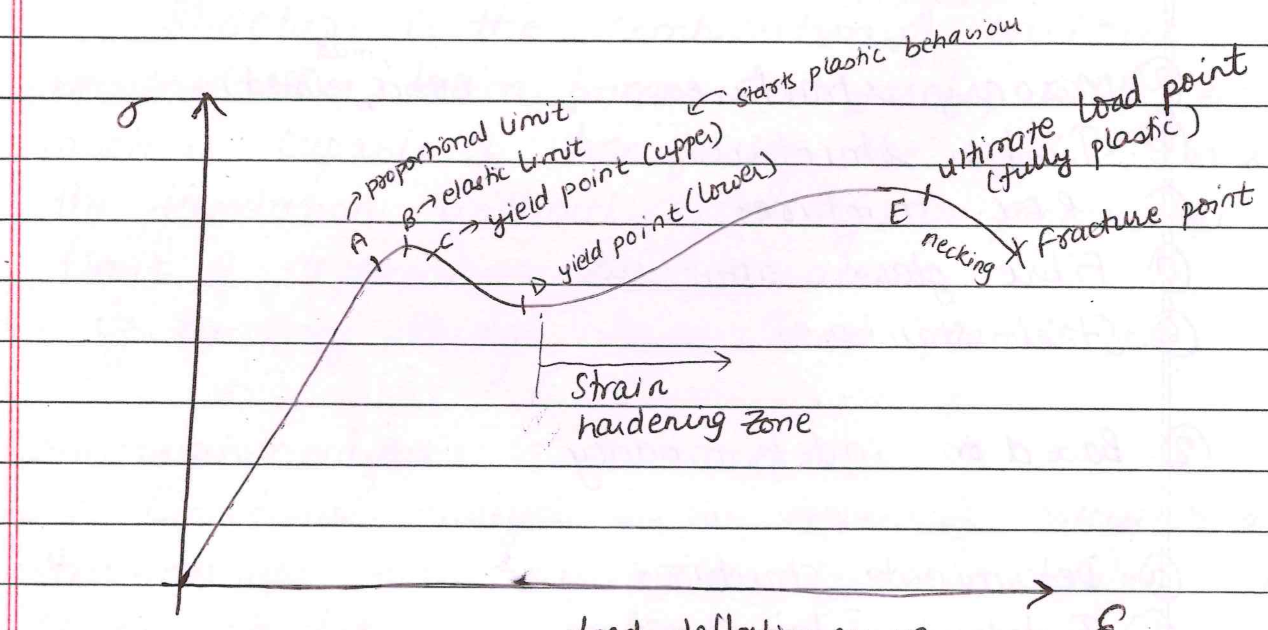
flexibility (δ) → displacement due to unit load
Stiffness (k) → force induced due to unit displacement

① Force method
② Displacement method

Force method	Displacement method
- Forces are treated as unknowns	- Displacements are treated as unknowns.
- Use compatibility equation to obtain ~~unknown~~ redundants	- Use eq^{n}s of static equilibrium to obtain unknown displacements
- Use eq^n of static equilibrium to obtain all reactions	- Use force displacement relationship to obtain unknown forces.
- Obtain internal stress parameters. $\underbrace{\Delta}_{\text{displacement}} = \dfrac{R \rightarrow \text{redundant}}{\delta \rightarrow \text{flexibility matrix}}$	$\overset{\text{force}}{F} = \underbrace{k}_{\text{stiffness}} \Delta \rightarrow \text{displacement}$

Linearly elastic structures:

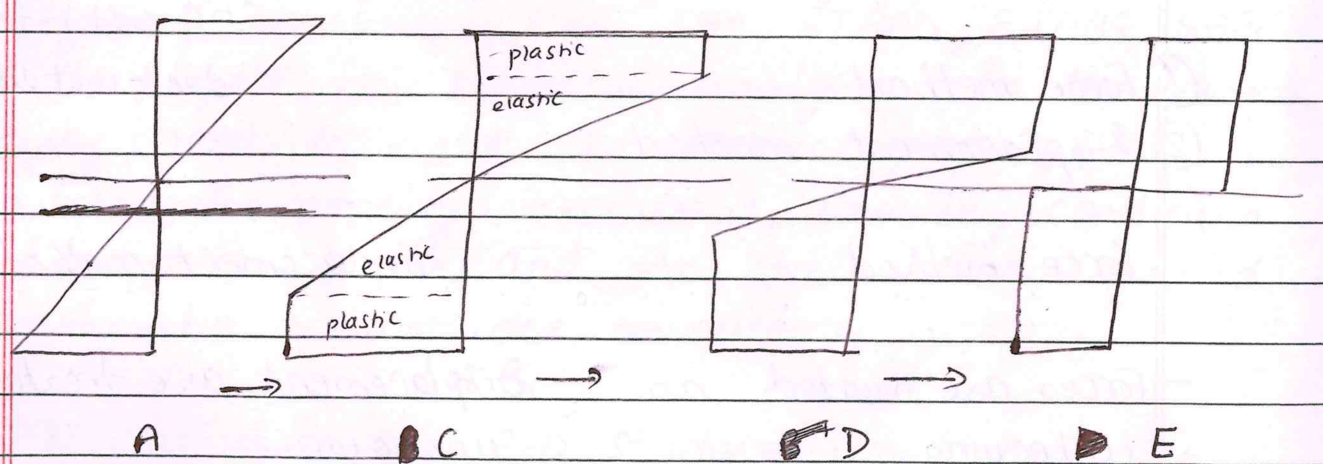
Structures whose stress-strain curve is linear (i.e. follows Hooke's law).



Load-deflection curve

For mild steel

Stress-strain curve ⇒ Force displacement curve (same)



Stress increasing →

Non-linearity in structural analysis:

Types:

1. Material non-linearity
2. Geometric non-linearity

Material non-linearity:

When material is stressed beyond it's proportional limit; it is no longer linear and material is strained beyond the yield point. Material non-linearity may affect the load deflection behaviour of structure even when the equilibrium equations for original geometry are still valid.

Geometric non-linearity:

If the load on the structure or it's deformation is large, then load deflection curve becomes non-linear. This is called geometric non-linearity. It may be caused due to large stress effect or large displacement effect. In such a case; we need to write equilibrium equations for deformed geometry.

Computer based methods of structural analysis:

1. Linear static method. (Load static & linear σ-ε curve)
2. Linear dynamic method. (Load dynamic but effect is linear curve)
   - Seismic coefficient method
   - Response spectrum method.
   - Mode superposition.

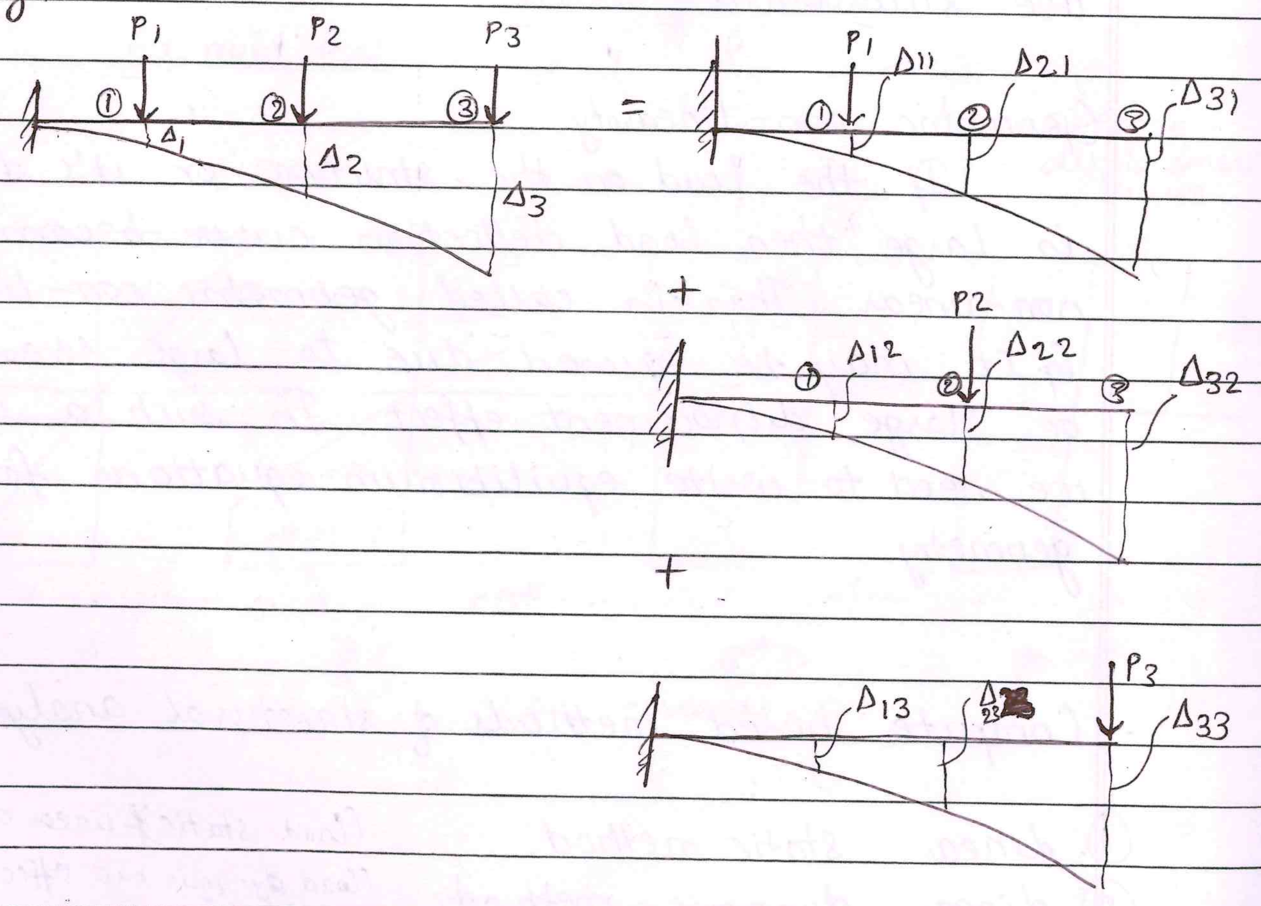
③ Non-Linear static method: (load are static load but $\sigma-\epsilon$ curve is non-linear)
- Push over analysis

④. Non-linear dynamic method: (Loads & $\sigma-\epsilon$ curve both are non-linear)
- Time history method

Principle of superposition:

The principle states that, "The displacements resulting from each of the number of forces may be added to obtain the displacements resulting from the sum of the forces." It is only valid for linear elastic structures which obey Hooke's law.

Eg



∴ Acc. to superposition theorem;

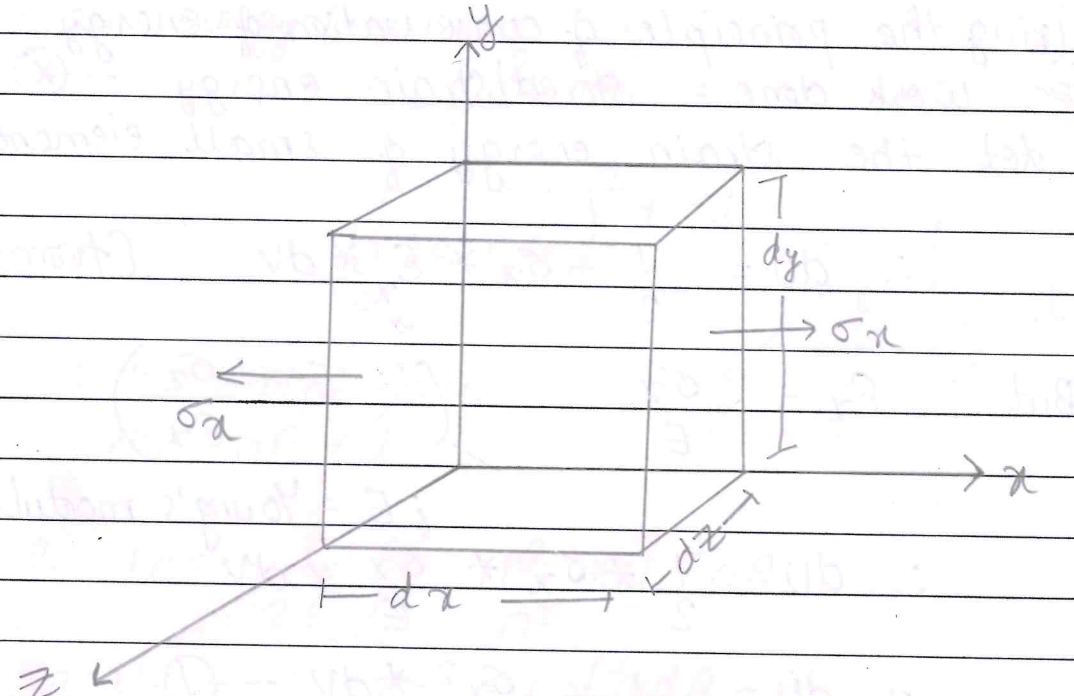
$$\Delta_1 = \Delta_{11} + \Delta_{12} + \Delta_{13}$$
$$\Delta_2 = \Delta_{21} + \Delta_{22} + \Delta_{23}$$
$$\Delta_3 = \Delta_{31} + \Delta_{32} + \Delta_{33}$$

where; Δ_{12} = Deflection at point 1 due to load P_2
--etc.

This principle is also valid for stresses and forces obeying Hooke's law.

CHAPTER - 2

STRAIN ENERGY METHOD



Let us consider an element of dimension $dx * dy * dz$ subjected to the stress σ_x along X-direction as shown in figure.

Then; the force on element due to stress σ_x is; $= \sigma_x * (dy * dz)$

Since the force is gradual (varies from zero to the maximum value);

the average is;

$$\text{Force}(F) = \overset{\text{initial (min)}}{0} + \overset{\text{max. (final)}}{\sigma_x * (dy * dz)}$$

$$\Rightarrow F = \frac{1}{2} * \sigma_x * dy * dz$$

($\because$ $\varepsilon_x = \frac{\text{deformation}}{\text{original length}}$)

Also; the deformation (or deflection) $= \varepsilon_x * dx$ (ε_x = strain in x-axis).

$\therefore$ work done = average force * deflection (or deformation)

$$= \frac{1}{2} * \sigma_x * dy * dz * \varepsilon_x * dx \quad \varepsilon_x \text{ as strain in x-axis}$$

$$= \frac{1}{2} * \sigma_x * \varepsilon_x * dV \quad (\because dx*dy*dz = dV)$$

Using the principle of conservation of energy;

Imp work done = stored/strain energy —(*)

Let the strain energy of small element be dU

$$\therefore dU = \frac{1}{2} * \sigma_x * \varepsilon_x * dV \quad \text{(from (1) rel}^n\text{)}$$

But $\varepsilon_x = \frac{\sigma_x}{E}$ $\left(\because E = \frac{\sigma_x}{\varepsilon_x}\right)$

; E = Young's modulus of elasticity

$$\therefore dU = \frac{1}{2} * \sigma_x * \frac{\sigma_x}{E} * dV$$

$$\Rightarrow dU = \frac{1}{2} * \frac{\sigma_x^2}{E} * dV \quad \text{—(1)}$$

Dividing both sides by dV;

$$\Rightarrow \boxed{\frac{dU}{dV} = \frac{1}{2} \cdot \frac{\sigma_x^2}{E}}$$

This is called strain energy density.

Now; total strain energy is obtained by integrating equation (1)

$$\int dU = \int \frac{1}{2} \frac{\sigma_x^2}{E} dV$$

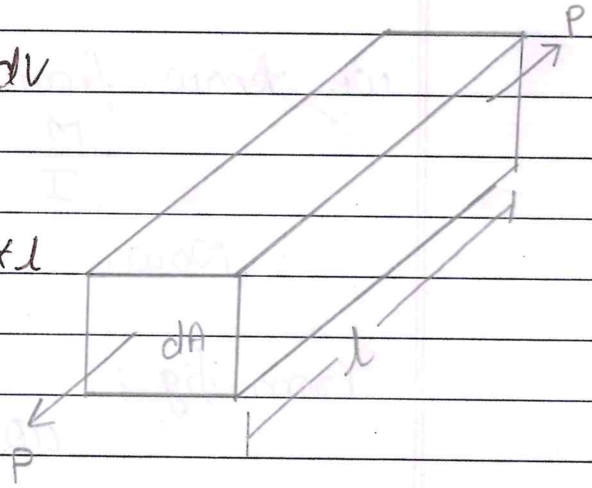
$$\Rightarrow \boxed{U = \int \frac{\sigma_x^2}{2E} dV} \quad \text{—(2)}$$

EXPRESSION FOR STRAIN ENERGY FOR :

(1) Bar member :

Let us consider a bar member of length 'l' and cross-section area 'dA'.

Now;



$$\text{strain energy } (U) = \int_0^l \frac{1}{2} \frac{\sigma_x^2}{E} dV$$

$$= \int_0^l \frac{1}{2E} \left(\frac{P}{A}\right)^2 dA * l$$

; P = axial force

dV = dA * l

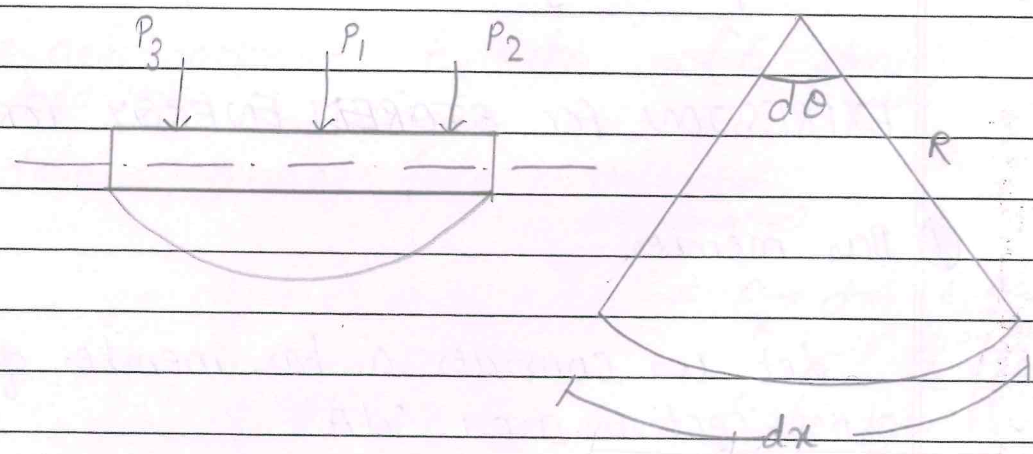
$$\Rightarrow U = \frac{1}{2E} \int_0^l \frac{P^2}{A^2} * l * dA$$

$$\Rightarrow U = \frac{1}{2E} \frac{P^2}{A^2} * l \int_0^l dA$$

$$\Rightarrow U = \frac{1}{2E} \frac{P^2}{A^2} * l * A$$

$$\Rightarrow \boxed{U = \frac{P^2 l}{2AE}}$$

Bending energy due to bending moment :



we know; from flexure theory;

$$\frac{M}{I} = \frac{\sigma}{y} = \frac{E}{R}$$

Now; $\sigma = \frac{M}{I} * y$

From fig;

$$d\theta = \frac{dx}{R} \qquad \left(\because \theta = \frac{l}{r}\right)$$

$$\Rightarrow R = \frac{dx}{d\theta} \quad \text{—(1)}$$

also; $\frac{M}{I} = \frac{E}{R}$ —(2)

From (1) & (2);

$$\frac{M}{EI} = \frac{d\theta}{dx}$$

$$\Rightarrow d\theta = \frac{M}{EI} dx$$

Here;

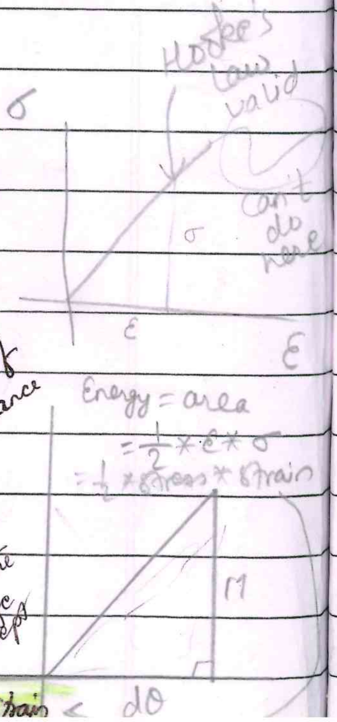
Strain energy $(dU) = \frac{1}{2} * \underset{\text{analogy of force}}{M} * \underset{\text{analogy of distance}}{d\theta}$

area of Δ

or same as average concept

$$\Rightarrow dU = \frac{1}{2} * M * \frac{M}{EI} dx$$

or ∴ energy $= \frac{1}{2} \times$ stress $\times$ strain

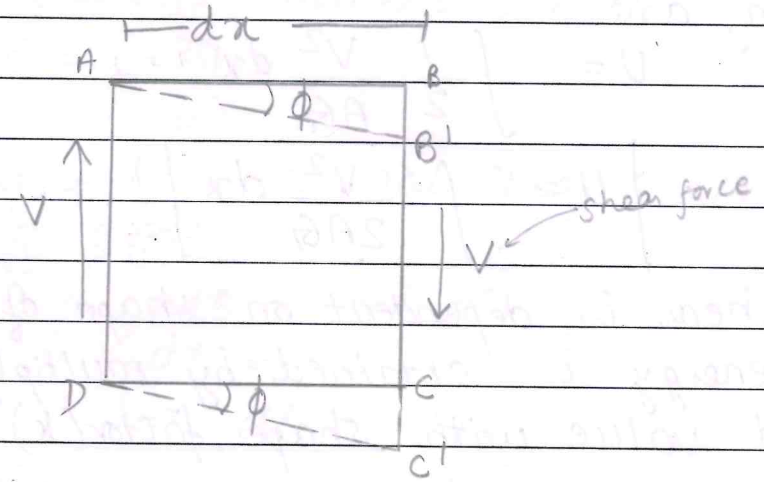


$$\Rightarrow dU = \frac{M^2}{2EI} dx$$

on integrating;

$$\boxed{U = \int_0^l \frac{M^2 dx}{2EI}}$$

Strain energy due to shear force :



Let us consider an elemental area ABCD subjected to shear force 'V' and gets deformed as shown in the figure.

Let

τ = shear stress

G = shear modulus.

Then; shear strain $(\phi) = \frac{\tau}{G} = \frac{V}{AG}$ $\left(\because \underset{\text{shear stress}}{(\tau)} = \frac{\overset{\text{shear force}}{V}}{A}\right)$

Again from figure; $\tan\phi = \frac{BB'}{AB}$

$$\Rightarrow \phi = \frac{BB'}{AB} \quad \text{(for small angles)}$$

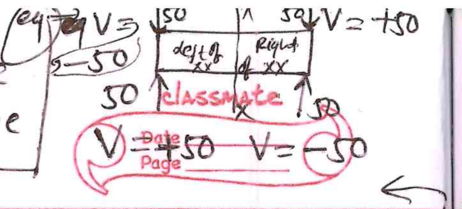
$$\Rightarrow BB' = \phi\, AB$$

$$\Rightarrow BB' = \phi\, dx$$

(Imp) Sign convention for shear forces (V):

deft ↑ then +ve   Right ↑ then −ve

deft ↓ then −ve   Right ↓ then +ve



Now; strain energy $(dU) = \frac{1}{2} * \text{stress} * \text{strain}$

$$\Rightarrow dU = \frac{1}{2} * V * \phi dx$$

$$\Rightarrow dU = \frac{1}{2} * V * \frac{V}{AG} dx$$

$$\Rightarrow dU = \frac{1}{2}\frac{V^2}{AG} dx$$

Total energy is obtained by integrating the above equation as:

$$U = \int \frac{1}{2}\frac{V^2}{AG} dx$$

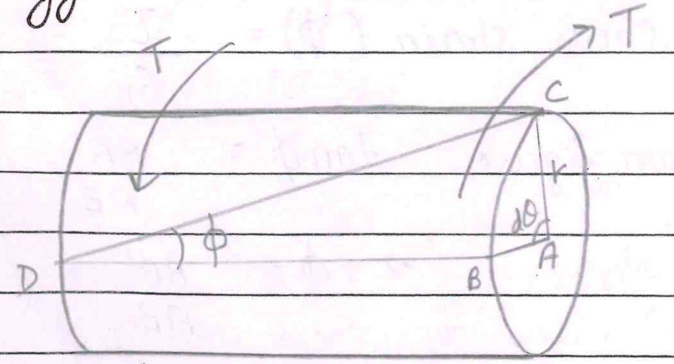
$$\Rightarrow \boxed{U = \int \frac{V^2 dx}{2AG}}$$

Since shear is dependent on shape of the section; strain energy is obtained by multiplying the standard value with shape factor(k)

i.e. $\boxed{U = k\int \frac{V^2 dx}{2AG}}$; k = Shape factor

k = 1.2 for rectangle
1.6 for circle
2.3 for triangle

Strain energy due to torsion:



Let us consider a shaft of radius 'r' subjected to torsion 'T' as shown in figure. From figure; torsion strain $= \frac{BC}{BD}$ (??)

$U_{Bending} = \int \frac{M^2 dx}{2EI}$ $U_{Shear} = k\int \frac{V^2 dx}{2AG}$ $U_{axial} = \frac{P^2 l}{2AE}$

Sign: sagging +ve, hogging −ve   Sign: PTO   Sign: −ve (Towards joint)   +ve (Away from joint)

Again; $d\theta = \frac{BC}{r}$

$$\Rightarrow BC = d\theta \cdot r$$

$\therefore$ torsion strain $= \frac{r d\theta}{dx}$ — (a)

again; torsion shear strain $= \frac{\tau}{G}$ — (b) ($\because$ strain $= \frac{\text{stress}}{\text{shear modulus of elasticity}}$)

also we know;

$$\frac{T}{J} = \frac{\tau}{r}$$

($\because \frac{T}{J} = \frac{\tau}{R}$) here.

; J = polar moment of inertia $= I_{xx} + I_{yy}$

$$\therefore \tau = \frac{T \cdot r}{J}$$

Keeping in (b); torsional shear strain $= \frac{T \cdot r}{GJ}$ — (c)

From (a) & (c);

$$\frac{\cancel{r} d\theta}{dx} = \frac{T \cdot \cancel{r}}{GJ}$$

$$\Rightarrow d\theta = \frac{T dx}{GJ}$$

Finally; work done $(dU) = \frac{1}{2} * \text{Stress} * \text{strain}$

$$\Rightarrow dU = \frac{1}{2} * T * d\theta$$

$$\Rightarrow dU = \frac{1}{2} * T * \frac{T dx}{GJ}$$

$\therefore$ Total strain energy $\boxed{(U) = \int \frac{1}{2}\frac{T^2 dx}{GJ}}$

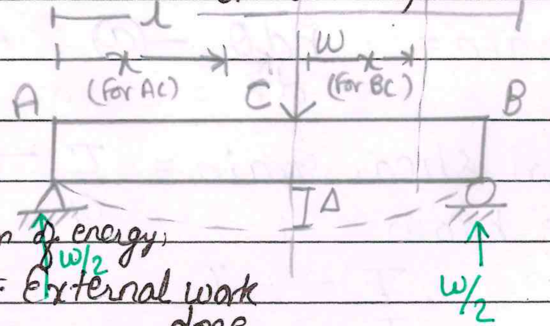
Note:
The total strain energy is obtained by summing up the strain energy due to A.F, S.F, B.M & torsion

Note (Imp)
For flexural member; strain energy due to Bending is more than 95% and strain energy due to other effects is less than 5%. So; Stored/Strain energy (S.E) due to bending only can be considered.

(Imp) if A — B — C in mid; then should break the segment to AB & BC segments.
if A — B not req^d, single segment

External work done $= \frac{1}{2} * w * \Delta$ (w → force, Δ → defl^n)

Q Calculate the deformation at the mid-point of simply supported beam due to point load 'w' at mid-span. Use real work method ~~Strain~~ and take constant flexure:



Δ = deflection

By law of conservation of energy;
Internal energy = External work done

$\Rightarrow U = \frac{1}{2} * w * \Delta$

But $U = \int \frac{M^2 dx}{2EI}$

; M = B.M. at any section

Calculation of rxn force:

$R_A = R_B = \frac{w}{2} (\uparrow)$

$M_{AC} = \frac{w}{2} * x$ (sag)

& $M_{CB} = \frac{w}{2} * x$ (sag)

$M = M_{AC} + M_{BC}$

Portion	origin	limit	M_x	EI
AC	A	0 to $l/2$	$\frac{w}{2} * x$	EI
CB	C	0 to $l/2$	$\frac{w}{2}(\frac{l}{2}+x) - wx$	EI

$\therefore U = \int_{AC} \frac{M^2 dx}{2EI} + \int_{CB} \frac{M^2 dx}{2EI}$

$\Rightarrow U = \int_0^{l/2} \left(\frac{wx}{2}\right)^2 \frac{dx}{2EI} + \int_0^{l/2} \left[\frac{w}{2}\left(\frac{l}{2}+x\right) - wx\right]^2 \frac{dx}{2EI}$

$\Rightarrow U = \frac{w^2}{8EI} \int_0^{l/2} x^2 dx + \frac{1}{2EI} \int_0^{l/2} \left(\frac{w^2 l^2}{16} - \frac{w^2 lx}{4} + \frac{w^2}{4} x^2\right) dx$

$\Rightarrow U = \frac{w^2 l^3}{192EI} + \frac{w^2 l^3}{64EI} - \frac{w^2 l^3}{64EI} + \frac{w^2 l^3}{192EI}$

$\Rightarrow U = \frac{w^2 l^3}{96EI}$

So; we know;
work done = Stored/strain energy

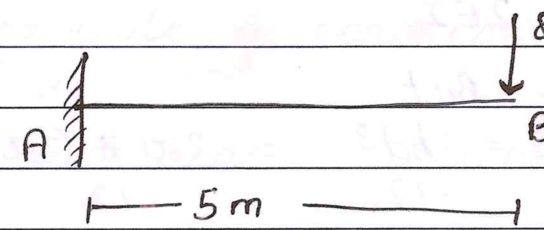
$\Rightarrow \frac{1}{\cancel{2}} * \cancel{w} * \Delta = \frac{w^{\cancel{2}} l^3}{\cancel{96}_{48} EI}$

$\Rightarrow \boxed{\Delta = \frac{w l^3}{48EI}}$

Here; calculated S.E. by bending only as 95% is by bending.
Add S.E. of shear and torsion as well if said and axial

Limitations of strain energy method:
→ Cannot be used if two forces (external loads) are given in question
→ Able to determine the deflection in only the place where the force (external loads) is given.

Q Calculate deflection at 'B' by considering shear only. Take $b * d = 250\ mm * 250\ mm$. $G = 1 * 10^9\ N/mm^2$, $E = 2 * 10^5\ N/mm^2$



Also calculate by considering bending only.

Here;
Shear Energy stored due to shear (U) $= \int K \frac{V^2 dx}{2AG}$

K - shape factor: Rectangle : 1.2
Circle : 1.6
Triangle : 2.3

$\angle V = +800$ KN (as sign convection PTO ~~next page~~)

$\therefore U = 1.2 * \int_{AB} \frac{V^2 dx}{2AG}$

$\Rightarrow U = 1.2 * \int_0^5 \frac{(800000)^2 dx}{2 * (250 * 250) * 10^9}$

$\Rightarrow U = 3.072$ J

External work done $= \frac{1}{2} * P * \Delta = \frac{1}{2} * 800 * \Delta$

Using principle of conservation of energy;

$\therefore 3.072 = \frac{1}{2} * 800 * \Delta$

$\Rightarrow \Delta = 7.68\ mm$

Strain energy due to bending :

$$U = \int \frac{M^2 dx}{2EI}$$

But

$$I = \frac{bd^3}{12} = \frac{250 * (250)^3}{12} = 3.255 * 10^{-4} m^4$$
$$= 325520833.3 \ mm^4$$

$$E = 2 * 10^5 \ N/mm^2$$

$\cancel{M = 800{,}000 x}$

$M = -800{,}000x$ | or $-4000000 + 800000x$

from right — from left

same result (can use any)

$$\therefore U = \int_0^{5000} \frac{(-800{,}000x)^2 dx}{2*(2*10^5)* 325520833.3}$$

$\Rightarrow U = 2*10237 5000 \ J$

$\cancel{\Rightarrow U = 3.072 \ J}$ $\cancel{\Rightarrow U = 255}$

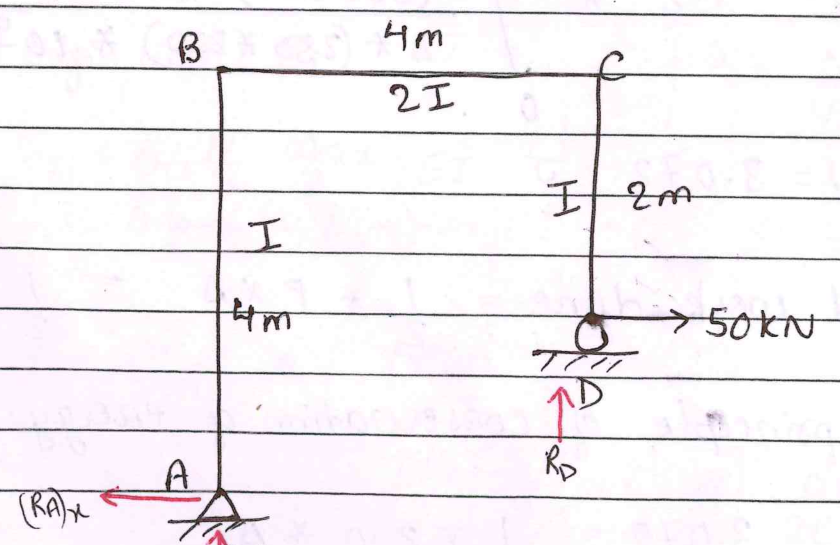
$\therefore \frac{1}{2} * 800 * 10^3 * \Delta = 2 * 10237 5000$

$\Rightarrow \Delta = 511.86 \ mm$

Q. Calculate the horizontal deflection at roller support (D) of frame as shown in figure. $E = 2*10^5 N/mm^2$

$A = 300 \ mm * 300 \ mm$

$G = 1 * 10^4 \ N/mm^2$



Consider all the effects due to bending, shear and axial loads.

Here :

Using principle of conservation of energy ;

External work done = Internal stored energy (U)

$= U_B + U_S + U_A$ — U_B: bending, U_S: shear force, U_A: axial force

Stored energy (S.E.) due to bending $(U_B) = \int \frac{M^2 dx}{2EI}$

$$U_S = k \int \frac{V^2 dx}{2AG}$$

$$U_A = \frac{\sum P^2 L}{2AE} = \int \frac{P^2 dx}{2AE}$$

Step - I :

Calculate reactions :

$\sum M_A = 0$ (↻ +ve)

$\Rightarrow 50 * 2 - R_D * 4 = 0$

$\Rightarrow R_D = 25 \ KN$ (↑)

(↑+ve) $\sum F_y = 0$

$\Rightarrow (RA)_y + R_D = 0$

$\Rightarrow (RA)_y = -R_D$

$\Rightarrow (RA)_y = -25 \ KN$ i.e. 25 (↓)

(+ve →) $\sum F_x = 0$

$\Rightarrow -(RA)_x + 50 = 0$

$\Rightarrow (RA)_x = 50 \ KN$ (←)

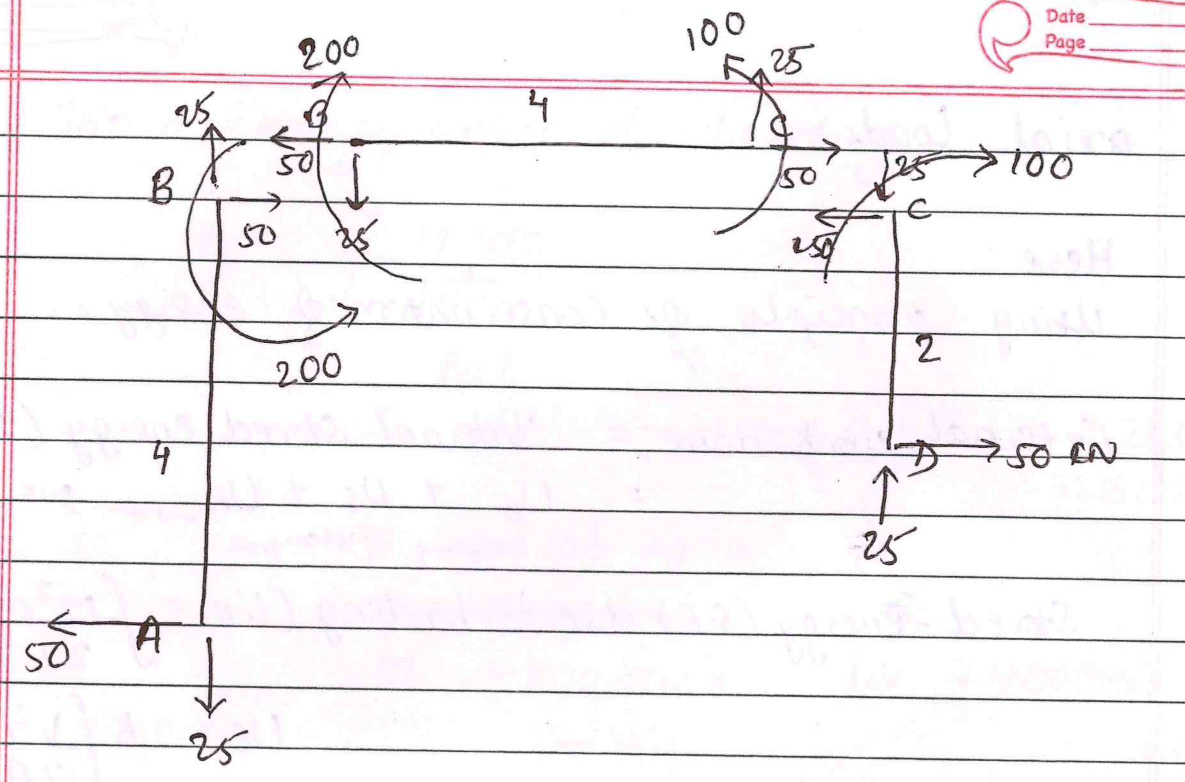


Table :

Portion	Origin	Limit	M_x	V	P	EI
AB	A	0 – 4	$50x$	+50	+25	EI
BC	B	0 – 4	$200-25x$	−25	+50	2EI
CD	C	0 – 2	$100-50x$	−50	−25	EI

Can take any origin or origin = C $100+25x$

$$U_B = \int \frac{M^2dx}{2EI} + \int_{BC} \frac{M^2dx}{2E(2I)} + \int_{CD} \frac{M^2dx}{2EI}$$

$$\Rightarrow U_B = \int_0^4 \frac{(50x)^2dx}{2EI} + \int_0^4 \frac{(200-25x)^2dx}{4EI} + \int_0^2 \frac{(100-50x)^2dx}{2EI}$$

$$\Rightarrow U_B = \frac{160000}{3EI}$$

Shear force :	Left ↑ then +ve	Right ↑ then −ve
Sign convention	Left ↓ then −ve	Right ↓ then +ve

$$U_S = K\int_{AB} \frac{V^2dx}{2AG} + K\int_{BC} \frac{V^2dx}{2AG} + K\int_{CD} \frac{V^2dx}{2AG}$$

$$= 1.2\left[\int_0^4 \frac{(50)^2dx}{2AG} + \int_0^4 \frac{(-25)^2dx}{2AG} + \int_0^2 \frac{(-50)^2dx}{2AG}\right]$$

$$= \frac{1.2}{2AG}\left[10000 + 2500 + 5000\right] = \frac{21000}{2AG}$$

$$U_A = \frac{\Sigma P^2 l}{2AE} = \left(\frac{P^2l}{2AE}\right)_{AB} + \left(\frac{P^2l}{2AE}\right)_{BC} + \left(\frac{P^2l}{2AE}\right)_{CD}$$

$$= \frac{(25)^2\cdot 4}{2AE} + \frac{(50)^2\cdot 4}{2AE} + \frac{(-25)^2\cdot 2}{2AE}$$

$$= \frac{13750}{2AE}$$

By work energy theorem ; work done = Energy stored

$$\Rightarrow \frac{1}{2} * W * \Delta = U_S + U_A$$

$$\Rightarrow \frac{1}{2} * 50 * \Delta = \frac{21000}{2AG} + \frac{13750}{2AE}$$

in Q/n given work

$$\Rightarrow 25\Delta = \frac{21000}{2*(300*300)*10^{-6}} + \frac{13750}{2*(300*300)*2*10^{8}}$$

$$\Rightarrow \Delta = 0.000482\ m \quad \because \text{in } KN/mm^2$$
(0.48 mm)

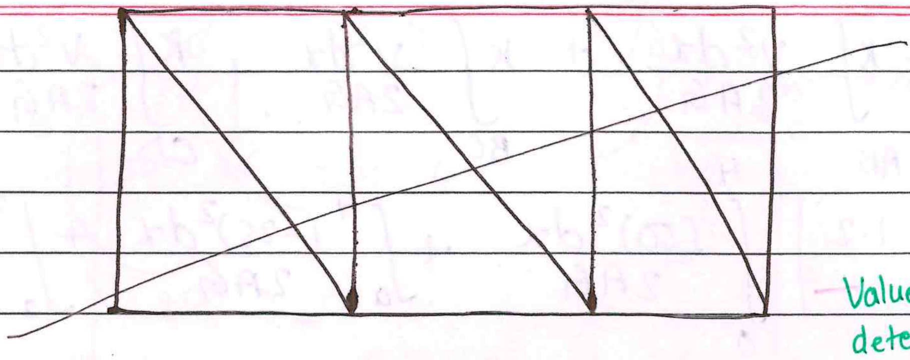
Q. Calculate the vertical deflection at 'G' of the given truss.

Take $E = 2*10^5\ N/mm^2$

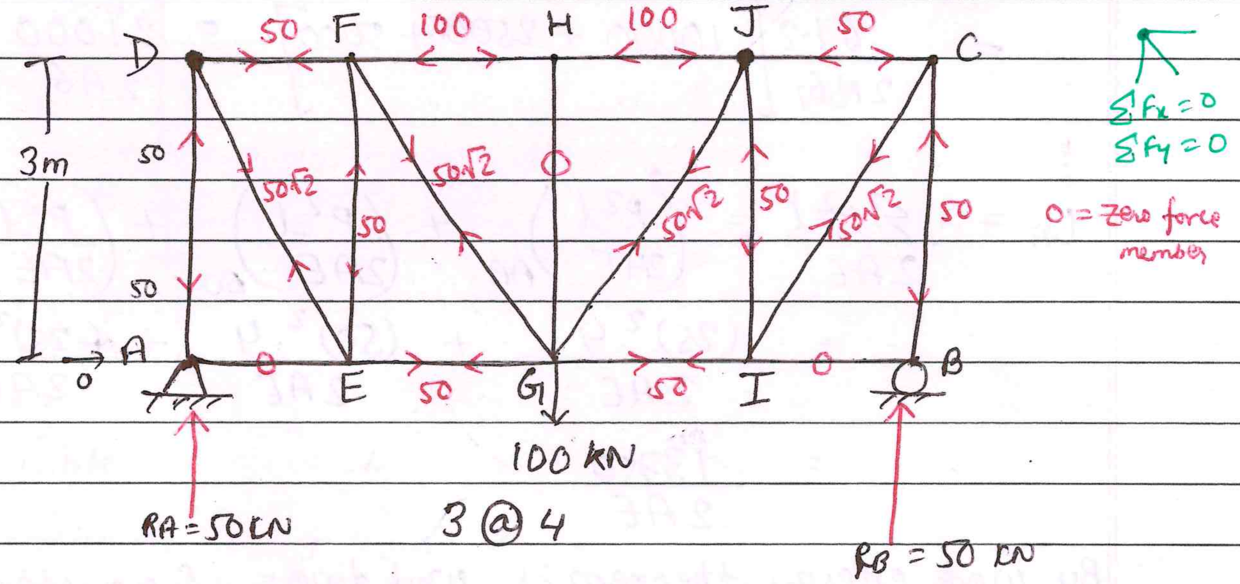
$\frac{l}{A}$ for vertical member = $6\ m^{-1}$

$\frac{l}{A}$ for horizontal member = $4\ m^{-1}$

$\frac{l}{A}$ for inclined member = $12\ m^{-1}$



Values 50, 100, $50\sqrt{2}$, etc determined by pin method



$$\Sigma M_A = 0 \quad (+ve \circlearrowright)$$

$$\Rightarrow 100 * 6 - R_B * 12 = 0$$

$$\Rightarrow R_B = 50 \text{ kN } (\uparrow)$$

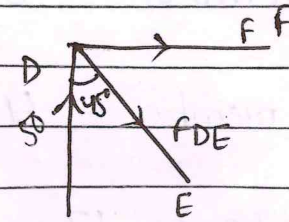
and $\qquad R_B + (R_A)_y = 100 \qquad (+\uparrow \Sigma F_y = 0)$

$$\Rightarrow (R_A)_y = 100 - 50$$

$$\Rightarrow (R_A)_y = 50 \text{ kN } (\uparrow)$$

$$(R_A)_x = 0 \quad (\because \xrightarrow{+ve} \Sigma F_x = 0)$$

At D :



$(\xrightarrow{+ve}) \Sigma F_x = 0$

$\Rightarrow F_{DF} + F_{DE} \sin 45° = 0$ — (1)

$(+ve) \Sigma F_y = 0$

$\Rightarrow 50 - F_{DE} \cos 45° = 0$

$\Rightarrow F_{DE} = 50\sqrt{2}$ Keeping in (1) ;

$F_{DF} = -50$

similarly for all

Sign convention for A.F	
Towards joint	$\Rightarrow -ve$
Away from joint	$\Rightarrow +ve$

Table :

Member	l/A	P	$\frac{P^2 l}{A}$
AE	4	0	0
EG	4	+50	10000
DF	4	−50	10000
FH	4	−100	40000
AD	6	−50	15000
EF	6	−50	15000
GH	6	0	0
DE	12	$50\sqrt{2}$	60000
FG	12	$50\sqrt{2}$	60000

$$\Sigma \frac{P^2 l}{A} = 2{,}10{,}000$$

for half truss ($\because$ symmetric) (for half symmetric)

$\therefore U_A = \dfrac{\Sigma P^2 l}{2AE}$ (Internal strain energy)

$= \dfrac{2{,}10{,}000}{2 * E} * 2$ for full (of formula)

$= 1.05 * 10^{-3}$

Also ;

Energy due to external work done $= \frac{1}{2} * 100 * \Delta$

$= 50\Delta$

$\therefore$ By conservation of energy ;

$$50\Delta = 1.05 * 10^{-3}$$

$$\Rightarrow \Delta = 2.1 * 10^{-5} \text{ m}$$

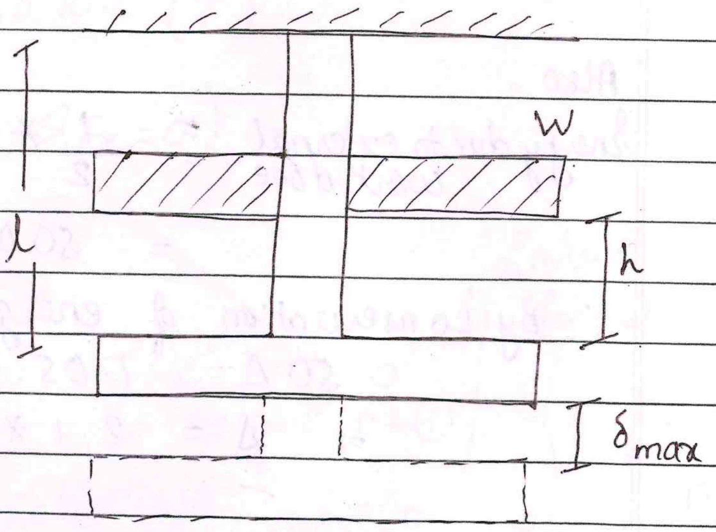
Strain Energy due to impact loads:

The loads which vary with time are called impact loads. Eg. load of freely falling objects, moving mass, etc. Analysis of such loads are done by the following assumptions:

1. The material behaves elastically and the no dissipation of energy takes place at the point of impact or at supports owing to the local inelastic deformation.
2. The inertia of a system resisting an impact load is neglected.
3. The deflection of a system is directly proportional to the magnitude of the applied force weather it is statically or dynamically applied.

Then; it is further assumed that the instant the moving body is stopped, kinetic energy is completely transferred to the internal ~~er~~ strain energy of resisting system. At that instant; maximum deflection occurs and vibration begins.

Derivation:



Let us consider a bar of length 'l' and cross-section area 'A' fitted to the collar as shown in figure. Let, W be the weight fall from height 'h' causing the deformation δ_{max} due to impact.

Let ;

- W – static load causing deflection δ_{st}.
- h – height of fall.
- δ_{st} – static deformation (due to W)
- δ_{max} – deformation due to impact (due to 'P')
- P – equivalent static / gradually applied load causing maximum deflection δ_{max}

we know;

$$\delta_{max} = \frac{Pl}{AE}$$

$$\delta_{st} = \frac{Wl}{AE}$$

Inside elastic limit;

$$\text{work done} = \frac{1}{2} * P * \delta_{max} \quad \text{—①}$$

also; work done by 'w' during the fall from height 'h'

$$= w * (h + \delta_{max}) \quad \text{—②}$$

final position

From ① & ②;

$$\frac{1}{2} * P * \delta_{max} = w * (h + \delta_{max})$$

$$\Rightarrow \frac{1}{2} * P * \frac{Pl}{AE} = w * \left(h + \frac{Pl}{AE}\right)$$

$$\Rightarrow \frac{P^2 l}{2AE} - \frac{wPl}{AE} - wh = 0$$

$\Rightarrow$ Dividing by $\frac{l}{2AE}$ on all side;

$$\Rightarrow P^2 - 2WP - 2AE \cdot \frac{Wh}{l} = 0$$

~~$\Rightarrow P^2 - 2WP$~~

$$\therefore P = \frac{2W \pm \sqrt{(2W)^2 - 4*1*\left(-2AE\frac{Wh}{l}\right)}}{2*1}$$

$$\Rightarrow P = W \pm \sqrt{W^2 + 2AE\frac{Wh}{l}}$$

$$\Rightarrow P = W\left[1 \pm \sqrt{1 + 2AE\frac{h}{Wl}}\right]$$

$$\Rightarrow \boxed{P = W\left[1 \pm \sqrt{1 + \frac{2h}{\delta_{st}}}\right]} \quad (3)$$

('−' removed as $P > W$ always)

Similarly;

$$\boxed{\delta_{max} = \delta_{st} * \left(1 \pm \sqrt{1 + \frac{2h}{\delta_{st}}}\right)} \quad (4)$$

Here; the term $\sqrt{1 + \frac{2h}{\delta_{st}}}$ is called impact factor. It might be enormously large for some cases of dynamic loads.

Corollary:

(1) When load is suddenly applied i.e. $h = 0$

$$\text{Impact factor (I.F.)} = 1 + \sqrt{1 + \frac{2h}{\delta_{st}}}$$

$$\therefore I.F. = 2$$

(2) ~~If $h << \delta_{st}$~~ If $h >> \delta_{st}$

$$I.F. = \sqrt{\frac{2h}{\delta_{st}}}$$

(3) If body is moving with certain velocity 'v' then; K.E. is completely transferred into strain energy.

$$K.E. = \frac{1}{2}mv^2$$

$$S.E. = \frac{1}{2} * P * \delta_{max}$$

Equating;

$$\frac{1}{2}mv^2 = \frac{1}{2}P\delta_{max}$$

$$\Rightarrow \frac{1}{2} \cdot \frac{W}{g} \cdot v^2 = \frac{1}{2} \cdot P \cdot \delta_{max}$$

As

$$P = \delta_{max}\frac{AE}{l} \quad \left(\because \delta_{max} = \frac{Pl}{AE}\right)$$

$$\therefore \frac{1}{2} \cdot \frac{W}{g} \cdot v^2 = \frac{1}{2} \cdot \delta_{max} \cdot \frac{AE}{l} \cdot \delta_{max}$$

$$\Rightarrow \frac{Wlv^2}{AEg} = \delta_{max}^2$$

$$\Rightarrow \delta_{max} = \sqrt{\frac{v^2}{g} \cdot \frac{Wl}{AE}}$$

$$\Rightarrow \delta_{max} = \sqrt{\frac{v^2}{g} \cdot \delta_{st}}$$

$$\Rightarrow \delta_{max} = \sqrt{\frac{v^2}{g} \cdot \frac{\delta_{st}^2}{\delta_{st}}}$$

$$\Rightarrow \boxed{\delta_{max} = \delta_{st}\sqrt{\frac{v^2}{g\delta_{st}}}}$$

Q. A weight of 40 KN ~~is~~ falls from a height of 100 mm in the bar of diameter 10 mm. The length of bar is 1.5 m. Calculate the maximum stress and deformation of the bar. $E = 2.1 \times 10^5 \ N/mm^2$

Here;

$$\delta_{max} = \frac{Pl}{AE}$$

But

$$P = W\left[1 + \sqrt{1 + \frac{2h}{\delta_{st}}}\right]$$

$$\delta_{st} = \frac{Wl}{AE} = \frac{40000 * 1.5}{\frac{\pi (0.01)^2}{4} * 2.1 * 10^{11}} \cancel{= 2.42 * 10^{-4} m} = 3.63 * 10^{-3} m$$

$$\therefore P = 40000\left[1 + \sqrt{\frac{2 * 0.1}{3.63 * 10^{-3}}}\right]$$

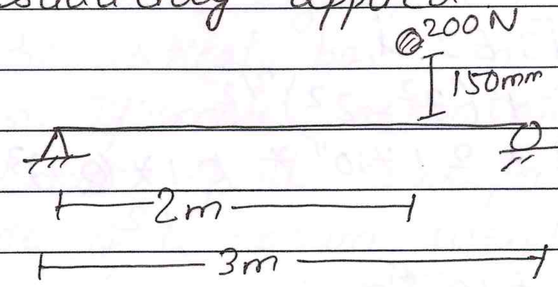
$$\Rightarrow P = 2239114.858 \ N$$

$$\therefore \delta_{max} = \frac{2239114.858 * 1.5}{\frac{\pi (0.01)^2}{4} * 2.1 * 10^{11}}$$

$$\Rightarrow \delta_{max} = 0.2 \ m$$

Q. Find the instantaneous ~~and~~ maximum deflection and bending stress for a $100 * 100 \ mm^2$ simply supported steel beam of 3 m span when it is strucked at a distance 2 m from the left support by 200 N weight falling from a height of 150 mm above the top of the beam. Also find the maximum deflection and bending stress when the load is suddenly applied. $E = 2.1 * 10^{11} \ N/m^2$

$$\boxed{\delta_{st} = \frac{wb(l^2 - b^2)^{3/2}}{9\sqrt{3} EI l}}$$



We know;

In simply supported beam;

$$\cancel{M = \frac{Pab}{l}}$$

$$\cancel{\Rightarrow M =}$$

~~But~~

$$\cancel{P = W\left[1 + \sqrt{1 + \frac{2h}{\delta_{st}}}\right]}$$

$$\cancel{\delta_{st} = \frac{Wl}{AE}} = \frac{200 * 3}{0.01 *}$$

$$M = \frac{Wab}{l} = \frac{200 * 2 * 1}{3} = 133.33 \ Nm$$

$$\therefore \frac{M}{I} = \frac{\sigma_{st}}{y}$$

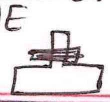
$$I = \frac{bh^3}{12} = \frac{0.1 * (0.1)^3}{12} = 8.33 * 10^{-6}$$

$$\Rightarrow \frac{133.33}{8.33 * 10^{-6}} = \frac{\sigma_{st}}{0.05}$$

$$\cancel{\Rightarrow \sigma_{st} = 799980}$$

$$\Rightarrow \sigma_{st} = 800300.12 \ N/m^2$$

use this in simply supported.
use $\delta_{st} = \frac{wl}{AE}$ in collar type



also;

$$\delta_{st} = \frac{wb\,(l^2-b^2)^{3/2}}{9\sqrt{3}\,EI\,l}$$

$$= \frac{200 * 1\,(3^2-2^2)^{3/2}}{9\sqrt{3} * 2.1*10^{11} * \frac{0.1*(0.1)^3}{12} * 3}$$

$$= 5.53 * 10^{-5} m$$

$$\therefore I.F. = 1 + \sqrt{1 + \frac{2h}{\delta_{st}}}$$

$$\Rightarrow I.F. = 1 + \sqrt{1 + \frac{2*0.15}{5.53*10^{-5}}}$$

$$\Rightarrow I.F. = 74.66$$

Now;

$$\delta_{max} = \delta_{st} * I.F.$$

$$\Rightarrow \delta_{max} = 5.53*10^{-5} * 74.66$$

$$\Rightarrow \delta_{max} = 4.128*10^{-3} m$$

for sudden impact;

$$I.F. = 2$$

$$\therefore \delta_{max} = 5.53*10^{-5}*2$$

$$\Rightarrow \delta_{max} = 1.106*10^{-4} m$$

also; $\sigma_{max} = I.F. * \sigma_{st}$.

$$\Rightarrow \sigma_{max} = 2 * 800300.12$$

$$\Rightarrow \sigma_{max} = 1600600.24 \; N/m^2$$

Q) An unknown weight falls through a height of 10 mm on a collar rigidly attatched to the lower end of the vertical bar 5 m long and 600 mm² is crossection. If max. instantaneous extension of the rod is 2 mm; find corresponding stress and magnitude of unknown weight. Take E = 200 GN/m².

Here;

$h = 10$ mm

$l = 5$ m

$A = 600 \; mm^2$

$\delta_{max} = 2$ mm

$E = 200 \; GN/m^2 = 200*10^9 \; N/m^2$

we know;

$$E = \frac{\sigma_{max}}{\epsilon_{max}}$$

$\Rightarrow$ But

$$\epsilon_{max} = \frac{\delta_{max}}{l} = \frac{2}{5000} = 4*10^{-4}$$

$$\therefore \sigma_{max} = E * \epsilon_{max}$$

$$\Rightarrow \sigma_{max} = 200*10^9 * 4*10^{-4}$$

$$\Rightarrow \sigma_{max} = 80000000 \; N/m^2$$

also;

$$\frac{1}{2} * P * \delta_{max} = w * (h + \delta_{max})$$

$$\cancel{\Rightarrow \frac{1}{2} * P * (2*10^{-3}) =}$$

$$\Rightarrow \frac{1}{2} * \left(\frac{\delta_{max} AE}{l}\right) \delta_{max} = w(h + \delta_{max})$$

$$\Rightarrow \frac{1}{2} \frac{(0.002)^2 * 600*10^{-6} * 200*10^9}{5} = w(0.01 + 0.002)$$

$$\Rightarrow w = 4000 \; N$$

Limitations of strain energy method:

① Strain energy method cannot be used if deformation is to be calculated at the point other than the point of load application.

② It can't be used if several loads are acting on the same structure.

CHAPTER - 3

ANALYSIS BY VIRTUAL WORK METHOD

The limitation of real work method is overcomed by virtual work method.

Consider a system of forces which is allowed by some virtual displacement. Then, the work done by real load due to the virtual displacement is called the virtual work. Virtual work is the product of following:

(i) Virtual work = Real load * Virtual deformation.

or (ii) Virtual work = Virtual load * Real deformation.

Virtual work method is based on the principle of virtual work.

※ Principle of virtual work for rigid body:

If a body is in equilibrium under a system of forces and is allowed some virtual deformation, then total virtual work done by

Virtual ~~load~~ work Method = Unit load Method = force method = Superposition method = Flexibility method = General method

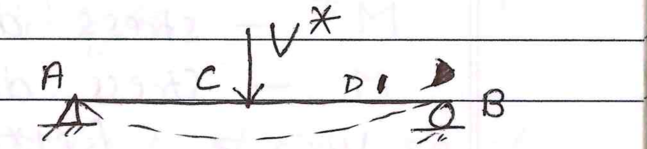
external forces is zero.

※ Principle of virtual work for deformable body:

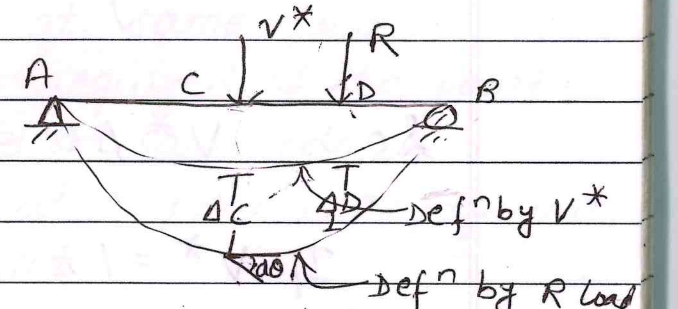
If a deformable body is under virtual 'Q' system of forces and remains in equilibrium under small displacement. Then, external virtual work done by 'Q' forces during the deformation is equal to the internal virtual work done by 'Q' stresses.

Let us consider a beam AB ~~such that~~ as shown in figure which is subjected to a virtual force V^* at 'C' (as shown in figure):

The deformation by V^* is shown through elastic curve.



Consider a real load 'R' at D causing deformation ΔC and ΔD as shown in figure at ~~respectively by loads~~ points ~~V^* and R.~~ 'C' & 'D'.



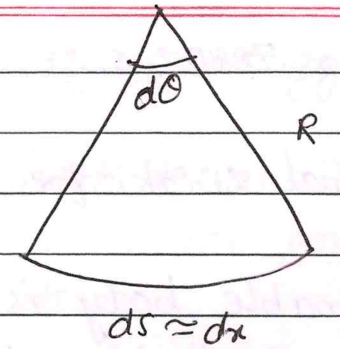
Using principle of virtual work at 'C'

External virtual work done at 'C' $= V^* \cdot \Delta C$ (virtual load, real deformation)

Internal virtual work done by stresses due to V^* during the deformation due to the real stresses (W_{int}).

In case of bending;

$W_{int} = M^* \cdot d\theta$; M^* - moment stress due to virtual load V^*



we know;

$$\frac{M}{I} = \frac{E}{R}$$

$$\Rightarrow d\theta = \frac{dx}{R} \qquad \left(\because \theta = \frac{l}{r}\right)$$

$$\Rightarrow \frac{1}{R} = \frac{d\theta}{dx}$$

$$\therefore \frac{M}{I} = \frac{E\,d\theta}{dx}$$

$$\Rightarrow d\theta = \frac{M}{EI}\,dx$$

M^* – stress due to virtual load V^*

M – Stress due to real load

$$\therefore W_{int} = \int M^* \left(\frac{M}{EI}dx\right)$$

So, $$V^* \Delta C = \int M^* \left(\frac{M\,dx}{EI}\right) \qquad \left\{ \because W_{int} = V^* \Delta C = M^* \Delta\theta \right.$$

If $V^* = 1$;

$$\Rightarrow 1 * \Delta C = \int M_1 \left(\frac{M\,dx}{EI}\right)$$

i.e. $$\boxed{\Delta C = \int M M_1 \frac{dx}{EI}}$$; one M is obtained by keeping real loads/forces (as of q/n) and next 'M' is by keeping unit.

M, M_0, M_1, ~~MM~~ are just notations.

Similarly; deformation due to shear $\Delta C,s = \int V_0 V_1 \frac{dx}{AG}$

deformation due to axial effect $\Delta C,a = \int F_0 F_1 \frac{dx}{AE}$

If all effects are present in a single ~~s~~ structure then total deflection is calculated by summing up the above:

$$\Delta = \Delta_b + \Delta_s + \Delta_a$$

Steps to analyze structure by virtual work method:

1. Calculate the reactions by using the equations of static equilibirium.
2. Calculate the bending moment in given system (M_0) (~~for~~ or primary system) due to given loads.
3. Apply unit load at the point and direction where deflection is desired and calculate the bending moment in unit load (M_1) system.

   ⓐ If vertical deflection is required at the point; apply unit vertical load at same point.
   ⓑ If horizontal deflection is required at the point; apply unit horizontal load at same point.
   ⓒ If rotation is required at a point; apply unit moment at the same point.
   ⓓ If angle of twist is required at a point; apply unit torsion at the same point.

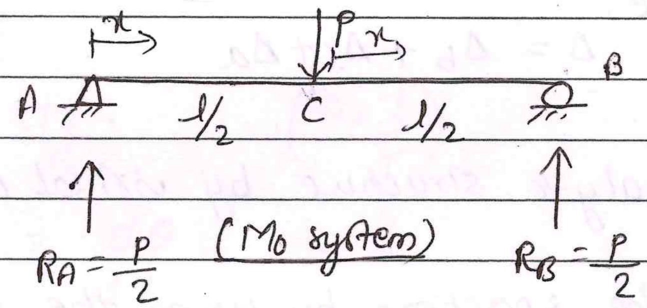
4. Calculate the deflections ~~by value~~ by using above expressions.

   ⓐ If obtained deflection is positive; then the direction for a no. of deflection req^{d} = same no. of unit loads at same point / apply V_A

is same as the direction of unit load.

(b) If obtained deflection is negative; then the direction is opposite as the direction of unit load.

Q. Calculate the deflection at 'C' and rotation at end 'A'.



Here;

Reactions : $R_A = R_B = \frac{P}{2}$

To get vertical deflection at 'C'; apply unit vertical load at 'C' and obtain M_1 system.

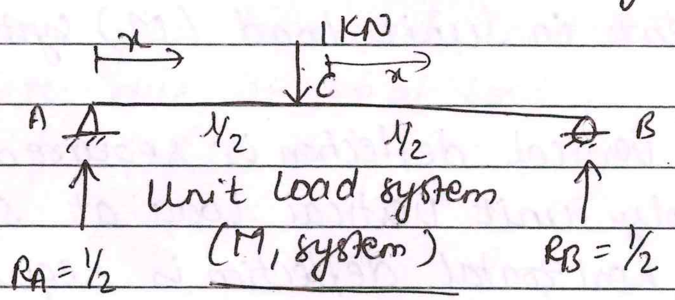


Table :

Portion	origin	Limit	M_0 at x (B.M. at 'x' in M_0 system)	BM at x (M_1) (B.M. at 'x' in M_1 system)	B.M. at x in M_2 system
AC	A	$0 - l/2$	$\frac{P}{2}x$	$\frac{x}{2}$	$\frac{1}{l}x - 1$
CB	C	$0 - l/2$	$\frac{P}{2}\left(\frac{l}{2}+x\right) - Px$	$\frac{1}{2}\left(\frac{l}{2}+x\right) - x$	$\frac{1}{l}\left(\frac{l}{2}+x\right) - 1$
or	B	$0 - l/2$	$\frac{P}{2}x$	$\frac{1}{2}x$	$-\frac{1}{l}x$

Same. Can use origin anywhere but sign convention should be followed (hog → -ve sag → +ve)

Now;

$$\Delta C = \int_{AC} M_0 M_1 \frac{dx}{EI} + \int_{BC} M_0 M_1 \frac{dx}{EI}$$

$$= \int_0^{l/2} \frac{P}{2}x * \frac{x}{2}\frac{dx}{EI} + \int_0^{l/2}\left[\frac{P}{2}\left(\frac{l}{2}+x\right) - Px\right]\left[\frac{1}{2}\left(\frac{l}{2}+x\right) - x\right]$$

same

$$= \int_0^{l/2} \frac{Px^2}{4}\frac{dx}{EI} + \int_0^{l/2}\left(\frac{P}{2}x * \frac{1}{2}x\right)dx$$

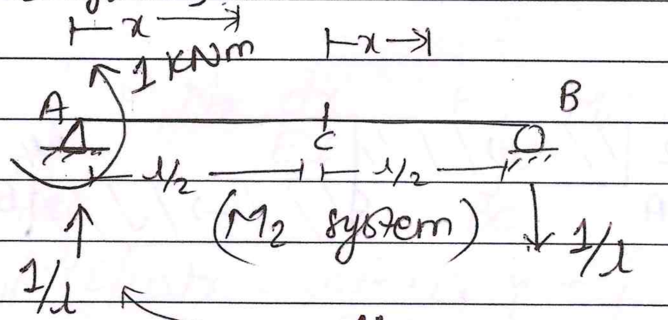
$$= 2\int_0^{l/2} \frac{Px^2}{4}\frac{dx}{EI}$$

$$= \frac{1}{2}\frac{P}{EI} \cdot \left[\frac{x^3}{3}\right]_0^{l/2}$$

$$= \frac{P}{2EI}\frac{l^3}{24}$$

$$= \frac{Pl^3}{48EI}$$

To get rotation at 'A'; remove all other loads and apply unit moment at all the other loads (M_2 system)



$\sum M_A = 1$ (↺ +ve)

$\Rightarrow R_B * l = 1$

$\Rightarrow R_B = \frac{1}{l}$

(↺ +ve) $\sum M_B = 0$

$\Rightarrow -1 + R_A * l = 0$

$\Rightarrow R_A = \frac{1}{l}$ (↑)

$$\therefore \theta_A = \int_{AC} M_0 M_2 \frac{dx}{EI} + \int_{BC} M_0 M_2 \frac{dx}{EI}$$

$$= \int_0^{l/2} \frac{P}{2}\cdot x\left(\frac{x}{l} - 1\right)\frac{dx}{EI} + \int_0^{l/2} \frac{P}{2}\cdot x\left(\frac{-x}{l}\right)\frac{dx}{EI}$$

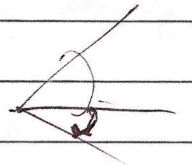
$$= \int_0^{l/2} \left(\cancel{\frac{Px^2}{2l}} - \frac{Px}{2} - \cancel{\frac{Px^2}{2l}} \right) \frac{dx}{EI}$$

$$= -\int_0^{l/2} \frac{Px}{2} \frac{dx}{EI}$$

$$= \frac{-P}{2EI} \left[\frac{x^2}{2} \right]_0^{l/2}$$

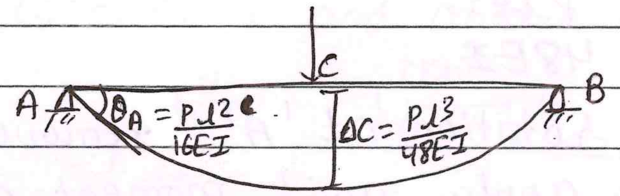
$$= \frac{-P}{2EI} \cdot \frac{l^2}{8}$$

$$= \frac{-Pl^2}{16EI}$$

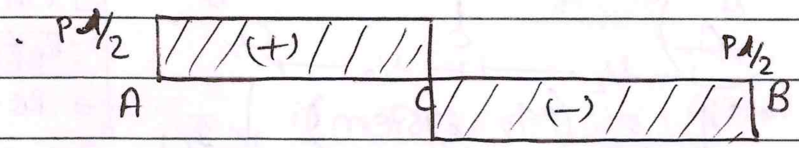


i.e. $\frac{Pl^2}{16EI}$

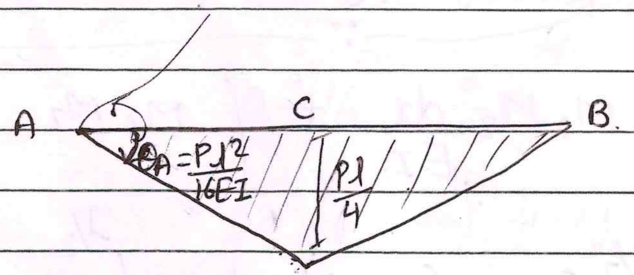
Elastic curve:



S.F.D.:

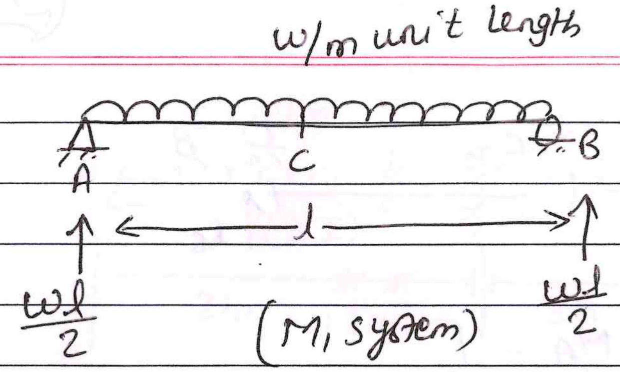


BMD



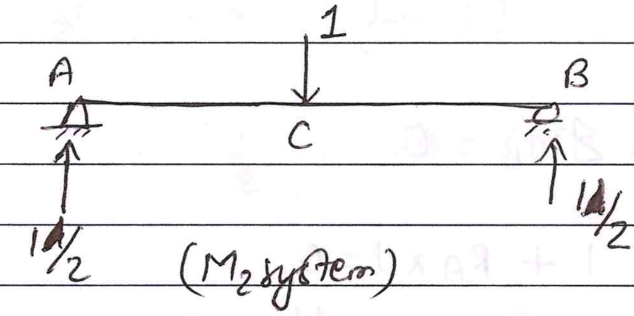
AFD.

Q



Find defn at C & slope at A & B

Here;



Portion	Origin	limit	B.M at M_1 system	BM at M_2 system	BM at M_3 system
AC	A	$0 - l/2$	$\frac{wlx}{2} - \frac{wx^2}{2}$	$\frac{x}{2}$	$-\frac{1}{l} \times x + 1$
CB	B	$0 - l/2$	$\frac{wlx}{2} - \frac{wx^2}{2}$	$\frac{x}{2}$	$\frac{1}{l} \cdot x$

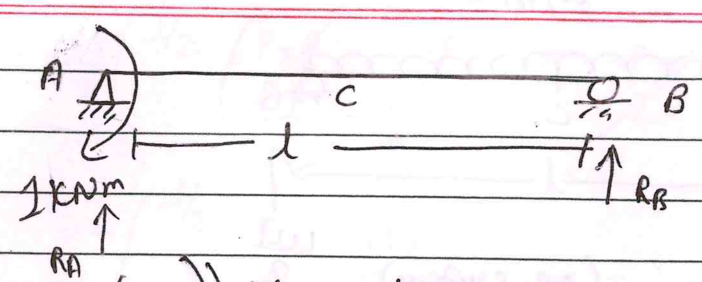
$$\Delta C = \int_{AC} M_1 M_2 \frac{dx}{EI} + \int_{BC} M_1 M_2 \frac{dx}{EI}$$

$$= \frac{1}{EI} \int_0^{l/2} \left(\frac{wlx}{2} - \frac{wx^2}{2} \right) \left(\frac{x}{2} \right) dx + \int_0^{l/2} \left(\frac{wlx}{2} - \frac{wx^2}{2} \right) \left(\frac{x}{2} \right) dx$$

$$= \frac{2}{EI} \int_0^{l/2} \left(\frac{wlx^2}{4} - \frac{wx^3}{4} \right) dx$$

$$= \frac{2}{EI} \frac{wl}{4} \cdot \frac{l^3}{32} - \frac{wl^4}{64}$$

$$= \frac{5wl^4}{384}$$



$(+ve)$ $M_A = 1$

$\Rightarrow -R_B * l = -1$

$\Rightarrow R_B = \frac{1}{l}$

and

$(+ve)$ $\Sigma M_B = 0$

$\Rightarrow 1 + R_A * l = 0$

$\Rightarrow R_A = -\frac{1}{l}$

Now;

$$\theta_A = \theta_B = \int_{ABC} M_1 M_3 \frac{dx}{EI} + \int_{BC} M_1 M_3 \frac{dx}{EI}$$

$$\Rightarrow \theta_A = \theta_B = 2\int_0^{l/2} \left(\frac{wlx}{2} - \frac{wx^2}{2}\right)\left(1 - \frac{x}{l}\right)\frac{dx}{EI} + \int_0^{l/2} \left(\frac{wlx}{2} - \frac{wx^2}{2}\right)\left(\frac{x}{l}\right)\frac{dx}{EI}$$

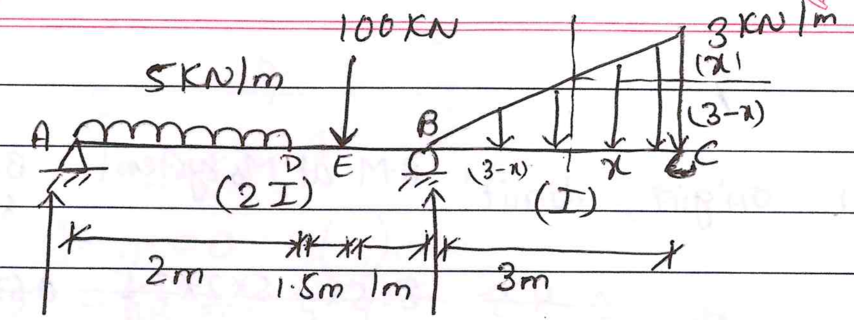
$$\Rightarrow \theta_A = \theta_B = \frac{1}{EI}\int_0^{l/2}\left(\frac{wlx}{2} - \frac{wx^2}{2}\right)dx$$

$$\Rightarrow \theta_A = \theta_B = \frac{1}{EI}\left(\frac{wl\,l^2}{16} - \frac{wl^3}{48}\right)$$

$$\Rightarrow \theta_A = \theta_B = \left(\frac{3wl^3 - wl^3}{48EI}\right)$$

$$\Rightarrow \theta_A = \theta_B = \frac{wl^3}{24EI}$$

(8)



$R_A =$ ~~86.5~~ 28 KN $\quad R_B =$ ~~28~~ 86.5 KN

(Given M_0 system)

Rxns:

$\Sigma M_A = 0$ $(+ve)$

$$\Rightarrow 5*2*\frac{2}{2} + 100*3.5 - R_B*4.5 + \frac{1}{2}*3*3*\left(4.5 + \frac{2}{3}*3\right) = 0$$

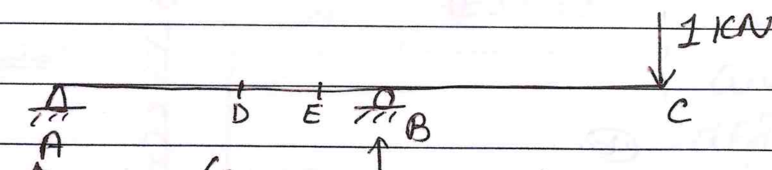
$\Rightarrow R_B = 86.5 KN$

$(\uparrow +ve)$ $\Sigma F_y = 0$

$\Rightarrow 86.5 - 5*2 - 100 + R_A - \frac{1}{2}*3*3 = 0$

$\Rightarrow R_A = 28$ KN

To calculate deflection at 'C'; remove all applied loads and apply 1 KN at 'C'



(Given M_1 system)

Rxns:

$\Sigma M_A = 0$ $(+ve)$

$\Rightarrow -R_B * 4.5 + 1 * 7.5 = 0$

$\Rightarrow R_B = 1.67$ KN

also; $R_A + 1.667 - 1 = 0$

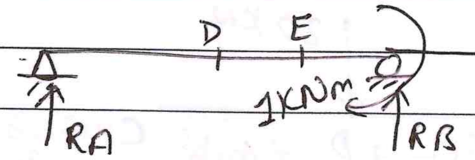
$\Rightarrow R_A = -0.67$ KN

Portion	origin	limit	B.M. at M_0 system	B.M. at M_1 system	B.M. at M_2 system
~~AB~~	~~A~~	~~0-4.5~~	~~86.5x - 5*2*3.5 -100*1~~	~~0.67*~~	
AD	A	0-2	$28x - 5*x*\frac{x}{2}$	$-0.67*x$	$-0.22x$
DE	D	0-1.5	$28(2+x) - 5*2*(1+x)$	$-0.67*(x+2)$	$-0.22(2+x)$
EB	B	0-1	$-\left(\frac{1}{2}*3*3\right)*\left(\frac{2}{3}\cdot 3+x\right) + 86.5x$	$-1(3+x) + 1.67x$	$-1+0.22x$
BC	C	0-3	$-\left(\frac{1}{2}*x*x*\frac{2}{3}x\right) - x*(3-x)*\frac{x}{2}$	$-1x$	0

$$\therefore \Delta C = \int_{AD} \frac{M_0 M_1\,dx}{2EI} + \int_{DE} \frac{M_0 M_1\,dx}{2EI} + \int_{EB} \frac{M_0 M_1\,dx}{2EI} + \int_{BC} \frac{M_0 M_1\,dx}{EI}$$

as given

To get rotation at 'B'; apply unit moment at 'B';



(B.M. at M_2 system)

$\Sigma M_A = 0$ (+ve)

$\Rightarrow -R_B * 4.5 + 1 = 0$

$\Rightarrow R_B = \frac{1}{4.5}$ KN $= 0.22$ KN

also;

(↑+ve) $\Sigma F_y = 0$

$\Rightarrow R_A + R_B = 0$

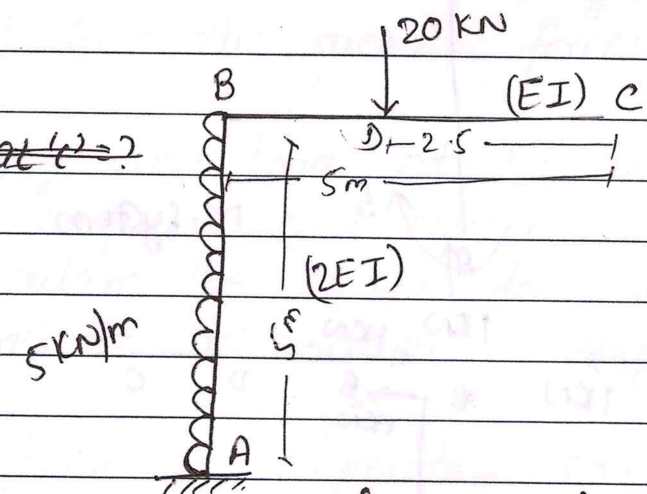
$\Rightarrow \frac{1}{4.5} + R_A = 0$

$\Rightarrow R_A = \frac{-1}{4.5}$ KN $= -0.22$ KN

Now;

$\theta_A = \ldots$

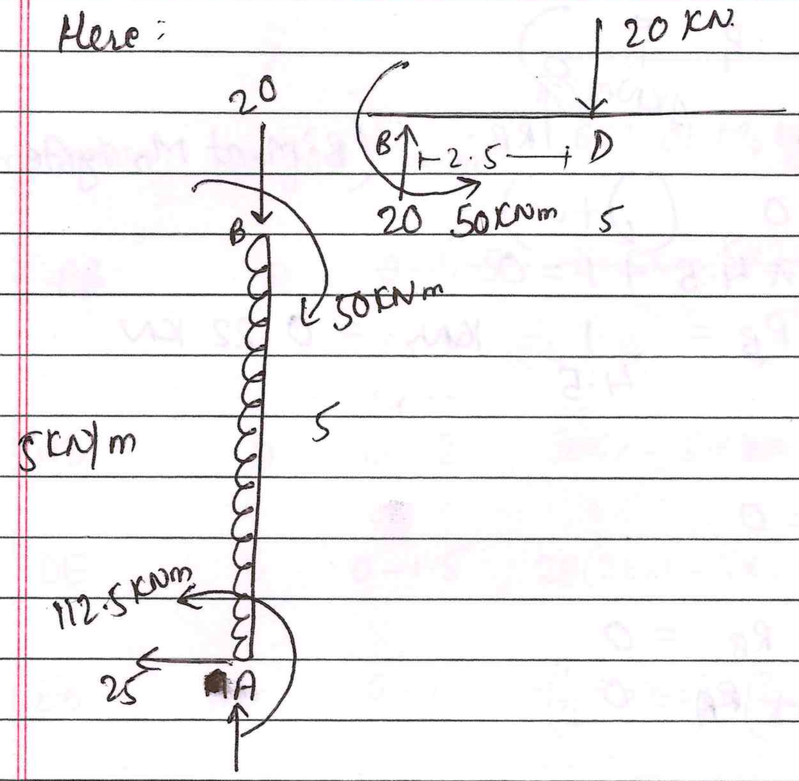
Q ~~Defn at 'C' = ?~~



(M_0 system)

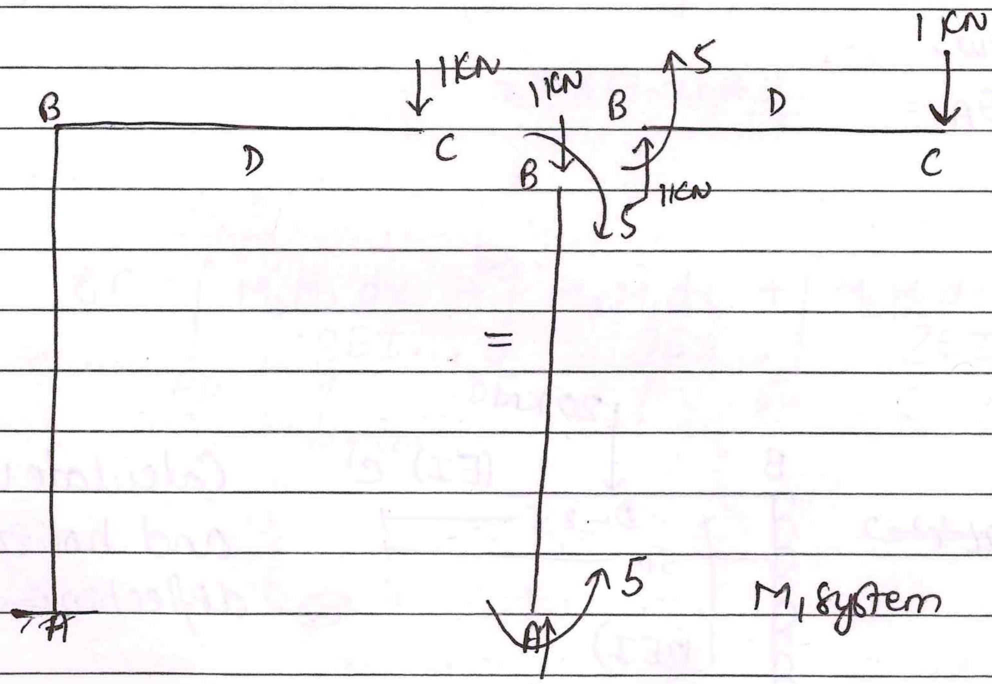
Calculate vertical and horizontal deflection at 'C'.

Here:



$M_A = 50 + 5 * 5 * 2.5$

$= 112.5$ (↺)



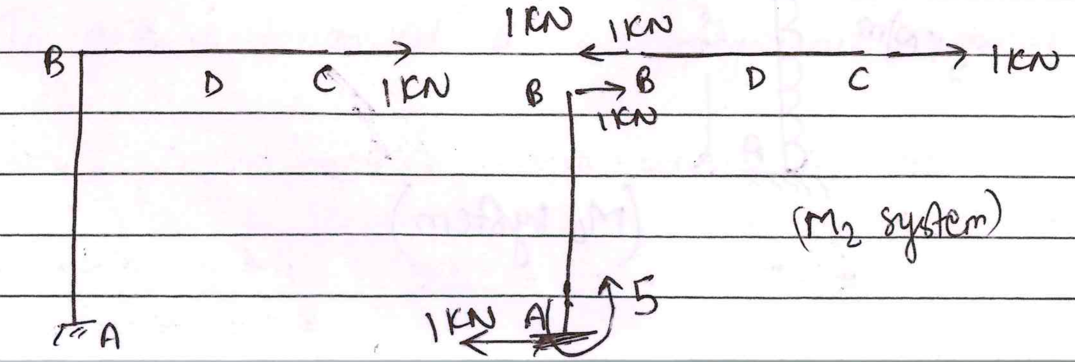


Table:

Portion	Origin	Limit	M_0	M_1	M_2	EI
AB	A	0-5	$-112.5 + 25x$ ~~2(5×5)~~ $- 5x \cdot x/2$	-5	$-5+x$	2EI
BD	B	0-2.5	$20 \cdot x - 50$	$-5 + 1 \cdot x$	0	EI
DC	C	0-2.5	0	$-1 \cdot x$	0	EI

Now;

$$\Delta C_{(vertical)} = \int_{AB} M_0 M_1 \frac{dx}{EI} + \int_{BD} M_0 M_1 \frac{dx}{EI} + \int_{DC} M_0 M_1 \frac{dx}{EI}$$

$= \dots$

$(\Delta C)_{Horizontal} = \dots$

Analysis of truss by virtual work method:

1. Calculate reactions by using equilibrium equations in given system
2. Calculate the member forces in given system i.e. F_0.
3. Apply unit load at the direction where deflection is required and obtain F_1 system.
4. Calculate the member forces in F_1 system.
5. Find the deformation by using $\sum \frac{F_0 F_1 l}{AE}$
6. If there is temperature change in any member; calculate:

$\delta_T = l\alpha(t_2 - t_1)$ $\Delta t = +ve$ for increasing temp. $\Delta t = -ve$ for decreasing temp.

If equal temp. rise in all section; no deflection arises

⑦. Calculate deflection due to deflection change as:

$$\Delta_T = F_1 * \delta_T$$

⑧ If there is lack of fit in any member i.e. δ_f;
sign convention:

too long $\Rightarrow$ +ve
too short $\Rightarrow$ −ve

Total deflection due to lack of fit $\boxed{\Delta_f = \delta_f * F_1}$

⑨. If the truss is subjected to all the effects of given load, temperature change and lack of fit; then total deflection is obtained by adding the deflection due to individual effects.

i.e. $\Delta = \Delta_L + \Delta_T + \Delta_f$

; $\Delta_L = \dfrac{\Sigma F_0 F_1 l}{AE}$

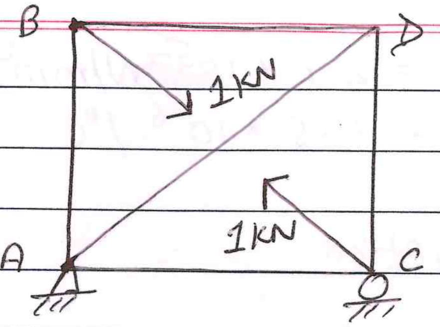
$\Delta_T = (l\alpha\Delta t) F_1$

$\Delta_f = \delta_f \cdot F_1$

Cases

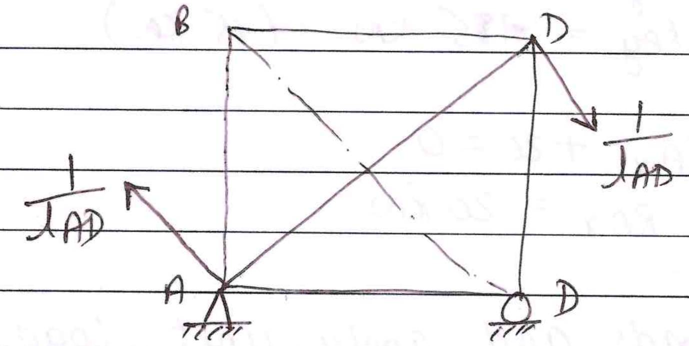
Case − I:

①. If relative deflection of any member (BC) is asked, apply unit load like this:



(no rxns induced at support by this unit load)

②. If rotation of any member AD is asked, apply the couple of magnitude $\dfrac{1}{l_{BC}}$ in the direction perpendicular to AD at joints 'A' and 'D' and obtain in unit load system as:



Q. Calculate the deflection at joint 'E' of the truss as shown in figure.

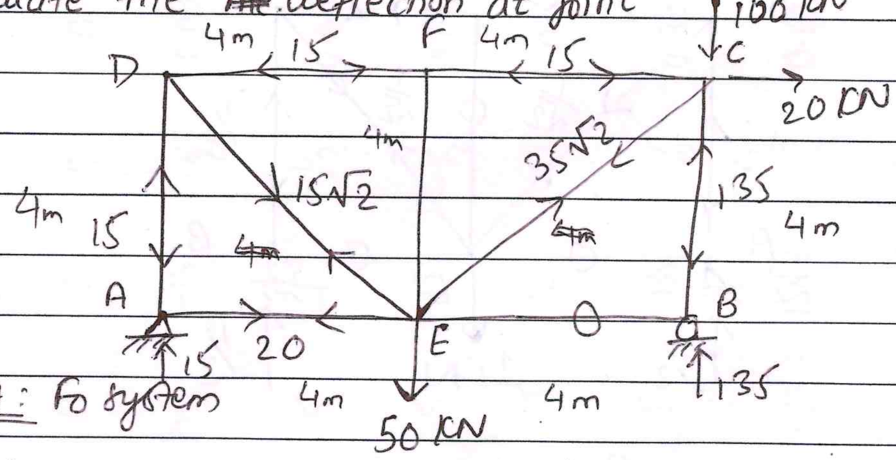


Fig: Fo system

Take Area of bottom chord members 1000 mm² and area of remaining members are 800 mm². Also calculate the deflection if the inclined members are subjected to rise in temperature by 25°C and horizontal members are 5 mm too long due to fabrication error. Take $E = 2.1 * 10^5$ N/mm^2

$\alpha = 11.5 * 10^{-6}$ /°C

Here;

Calculation of reaction

$\Sigma M_A = 0$ (↻ +ve)

$\Rightarrow 50*4 + 20*4 + 100*8 - R_B * 8 = 0$

$\Rightarrow R_B = 135$ KN

also; (↑+ve) $\Sigma F = 0$

$\Rightarrow R_{Ay} - \overset{-150}{100} + R_B = 0$

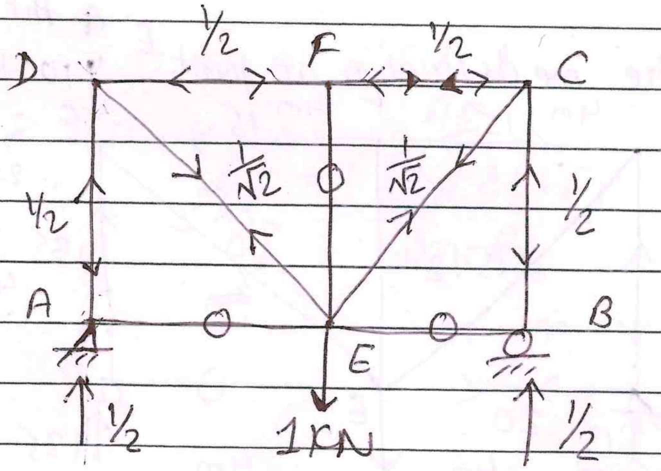
$\Rightarrow R_{Ay} - 150 + 135 = 0$

$\Rightarrow R_{Ay} = \cancel{-185}$ KN. $\leftarrow$ (15 KN)

and $-R_{Ax} + 20 = 0$

$\Rightarrow R_{Ax} = 20$ KN.

Remove all loads and apply unit load at joint E.



F_1 system

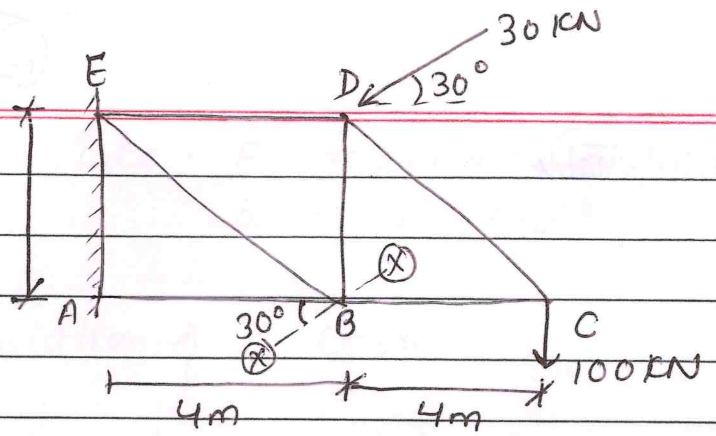
~~Table~~

~~Member~~

Table

Member	l (m)	A (mm^2)	$1/A$ ($*10^{-3}$)	P_0	P_1	$P_0 P_1 l/A$	t	$\delta_t = l\alpha t$	$\Delta T = \delta_t \cdot F_1$	δ_f ($*10^{-3}$)	$\Delta f = \delta_f \cdot F_1$
AE	4	1000	4	20	0	0				+5	0
EB	4	1000	4	0	0	0				+5	0
DF	4	800	5	−15	$-1/\sqrt{2}$	0.0375				+5	
FC	4	800	5	−15	$-1/2$	0.0375				+5	
AD	4	800	5	−15	$-1/2$	0.0375					
EF	4	800	5	0	0	0					
BC	4	800	5	−135	$-1/2$	0.3375					
DE	$4\sqrt{2}$	800	$5\sqrt{2}$	$15\sqrt{2}$	$1/\sqrt{2}$	0.1061	+25	0.00163	~~0.0345~~ 0.0115		
EC	$4\sqrt{2}$	800	$5\sqrt{2}$	$35\sqrt{2}$	$1/\sqrt{2}$	0.2475	+25	0.00163	~~0.0345~~ 0.0115		

H/W ⑧



Calculate deflected at x-x direction as indicated in figure.

- AB, BC ⇒ lack of fit 3 mm too short.
- BD ⇒ −25°C

Effect of temperature change in flexural members:

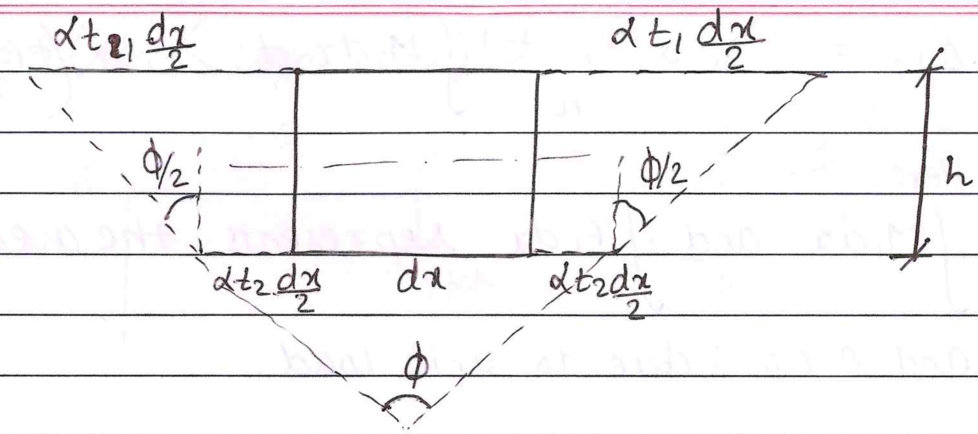
General formula for deflection at any direction 'n' (1) i.e. due to load.

$$\Delta_{1L} = \sum_1 \int M_1 M_0 \frac{dx}{EI} + \sum_1 \int F_0 F_1 \frac{dx}{AE} + \sum_1 \int V_1 V_0 \frac{dx}{AG}$$

Due to temperature change of flexural member will be subjected to considering rotation and axial deformation but the shear deformation is negligible. So; we can neglect the third term of above equation.

$$\Delta_{1T} = \sum_1 \int \overset{\text{moment rotation}}{M_1 M_T \frac{dx}{EI}} + \sum_1 \int F_1 F_T \frac{dx}{AE}$$

Let us consider a beam element dx subjected to rise in temperature by 't_1' and 't_2'°C at top and bottom fiber respectively.



Here;

$\phi = \frac{M_T\, dx}{EI}$ = mutual rotation of an element between two faces due to temperature change

$\Delta x = \frac{F_T\, dx}{AE}$ = linear expansion of an element due to temperature rise

$$\Delta_{1T} = \sum_1 \int M_1 \phi + \sum_1 F_1 \Delta x$$

where; from figure;

$$\tan\frac{\phi}{2} \simeq \frac{\phi}{2} = \frac{\left(\alpha t_1 \frac{dx}{2} - \alpha t_2 \frac{dx}{2}\right)}{h}$$

$$\Rightarrow \phi = \frac{\alpha(t_1 - t_2)}{h} dx$$

The expansion in the neutral axis; $\Delta x = \frac{\alpha t_1 dx + \alpha t_2 dx}{2}$

$$\Rightarrow \Delta x = \frac{\alpha(t_1 + t_2) dx}{2}$$

Substitute ϕ and Δx in above equation; we get:

$$\Delta_{1T} = \sum_1 \int M_1 \alpha \frac{(t_1 - t_2)}{h} dx + \sum_1 F_1 \frac{\alpha(t_1 + t_2) dx}{2}$$

$$\Delta_{1T} = \sum \alpha \frac{(t_1-t_2)}{h}\int M_1\,dx + \sum \alpha\left(\frac{t_1+t_2}{2}\right)\int F_1\,dx$$

Here ;

$\int M_1 dx$ and $\int F_1 dx$ represents the area of B.M.D. and A.F.D. due to unit load.

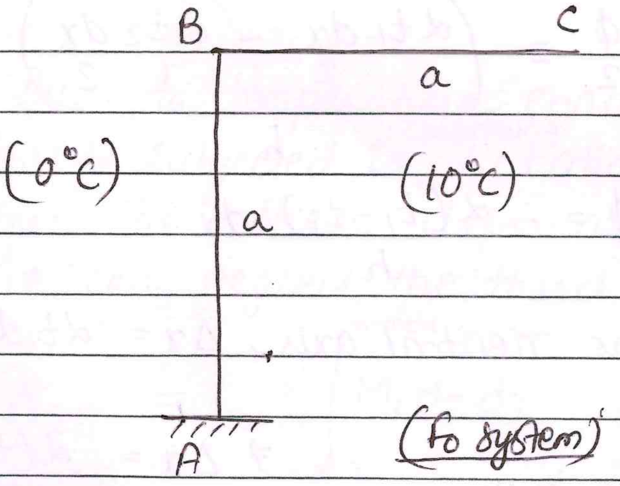
Sign convention:

Bending | Axial

① Unit load sagg ⇒ , Temp^r ⇒ ~~te~~ bottom tension $\}$ +ve | Unit load ⇒ Tension , Temperature expand ⇒ $\}$ +ve

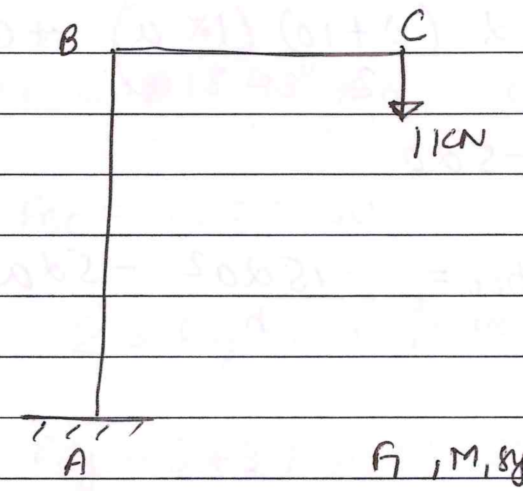
② $|t_1 - t_2|$ ⇒ always take absolute value.

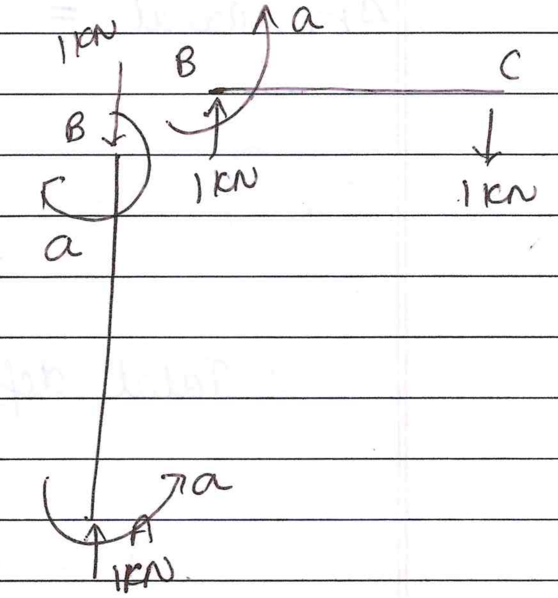
Q.

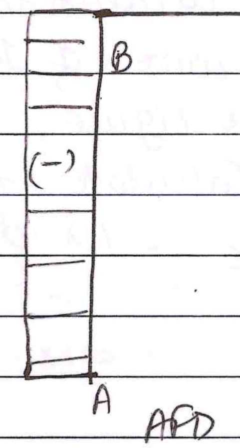


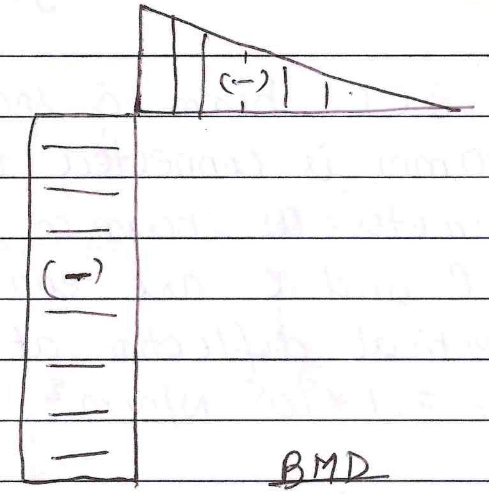
Calculate vertical deflection at 'C'.

F1 system (Unit load system):









Due to bending :

$$\Delta_{1T\,bending} = \left(\frac{\alpha(t_1-t_2)}{h}\int M_1\,dx\right)_{AB} + \left(\frac{\alpha(t_1-t_2)}{h}\int M_1\,dx\right)_{BC}$$

$$\Rightarrow \Delta_{1T} = -\left(\alpha\frac{|0-10|}{h} * (a*a)\right) - \left(\alpha\frac{|0-10|}{h} * \left(\frac{1}{2}*a*a\right)\right)$$

$$\Rightarrow \cancel{\Delta_{1T}\ axial} = \frac{-10\alpha a^2}{h} - \frac{5\alpha a^2}{h}$$

$$\Rightarrow \Delta_{1T} = \frac{-15\alpha a^2}{h}$$

$$\Delta_{1T,\,axial} = \Delta_{1T,\,AB} + \underbrace{\Delta_{1T,\,BC}}_{\text{no A.F.}}$$

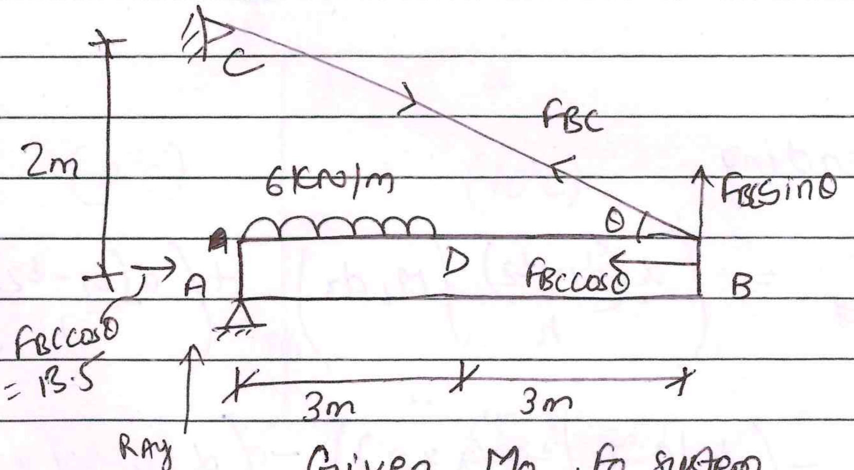
$$= -\alpha\left(\frac{(0+10)}{2}(1*a)\right) + 0$$

$$= -5\alpha a$$

$$\therefore \text{Total deflection} = \frac{-15\alpha a^2}{h} - 5\alpha a$$

Virtual work method for Hybrid structures:

Q) A R.C.C. beam of span 6m and dimension 200mm* 400mm is connected to a steel wire of 10 mm diameter as shown in figure. In the figure; joints A, B and C are end-connected. Calculate the vertical deflection at 'D'. Take $E_s = 2.1*10^5\ N/mm^2$; $E_c = 1*10^5\ N/mm^2$



Given M_0, F_0 system

$$\theta = \tan^{-1}\left(\frac{2}{6}\right) = 18.43^\circ$$

For rxns:

$$\sum M_A = 0$$

$$\Rightarrow -F_{BC}\sin 18.43^\circ * 6 + \frac{6*3^2}{2} = 0 \quad (\circlearrowleft +ve)$$

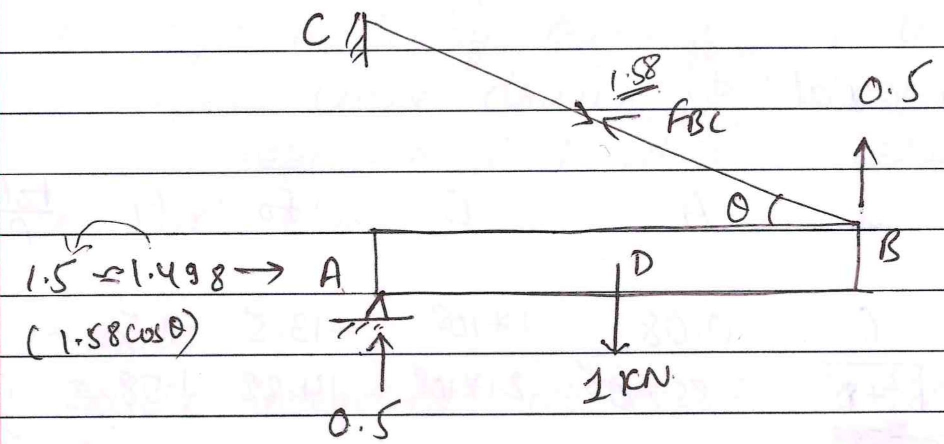
$$\Rightarrow F_{BC} = 14.23\ KN$$

Also; $\sum F_y = 0$ $(\uparrow +ve)$

$$\Rightarrow R_{Ay} - 6*3 + 14.23*\sin 18.43 = 0$$

$$\Rightarrow R_{Ay} = 13.5\ KN$$

For M_1, F_1 system:



$$F_{BC}\sin 18.43 = 0.5$$

$$\Rightarrow F_{BC} = 1.58$$

~~Also~~

Now;

$$\Delta D = \Delta D_{bending} + \Delta D_{axial}$$

$$I_{rod} = \frac{\pi d^4}{64} = \frac{\pi (10*10^{-3})^4}{64} = 4.91*10^{-10}\ m^4$$

$$I_{beam} = \frac{0.2*(0.4)^3}{12} = 1.06*10^{-3}\ m^4$$

$$A_{beam} = 0.2*0.4 = 0.08\ m^2$$

$$A_{rod} = \frac{\pi(10*10^{-3})^2}{4} = 7.85*10^{-5}\ m^2$$

Table for bending:

Portion	origin	Limit	M_0	M_1
AD	A	0-3	$13.5x - \frac{6x^2}{2}$	$0.5x$
DB	B	0-3	$\cancel{14.23\sin}$ $4.498\,x$ ($14.23\sin 18.43^\circ$)	$0.5x$

$$\Delta D \text{ bending} = \int_{AB} \frac{M_0 M_1\, dx}{EI} + \int_{DB} \frac{M_0 M_1\, dx}{EI}$$

Table for axial:

Member	l	A	E	F_0	F_1	$\frac{F_0 F_1 l}{AE}$
AB	6	0.08	$1*10^8$	-13.5	-1.5	
BC	$\sqrt{6^2+2^2}$ ($\cancel{7.21}$ = 6.32)	$7.85*10^{-5}$	$2.1*10^8$	14.23	1.58	

$$\therefore \Delta_{axial} = \sum \frac{F_0 F_1 l}{AE} \quad --$$

Limitations of virtual work method:

1. This method is too lengthy and tedious
2. Deflection at only one direction can be calculated at a time
3. Deflection is known; only if all force components of whole structural system is known

Chapter-4
Deflection in Beam

Deflection in beam is generally due to application of loads. There is also deflection due to lack of fit, settlements of supports, temperature variation, etc. Deflection in the structures should be within a permissible limit. So; it is necessary to calculate the deflection.

When a beam is subjected to a set of loads; it's axis bends. Such deflected shape is called deflection curve or elastic curve.

Slope at any point in a beam is defined as the angle made by original axis to the tangent of the elastic curve drawn through that point.

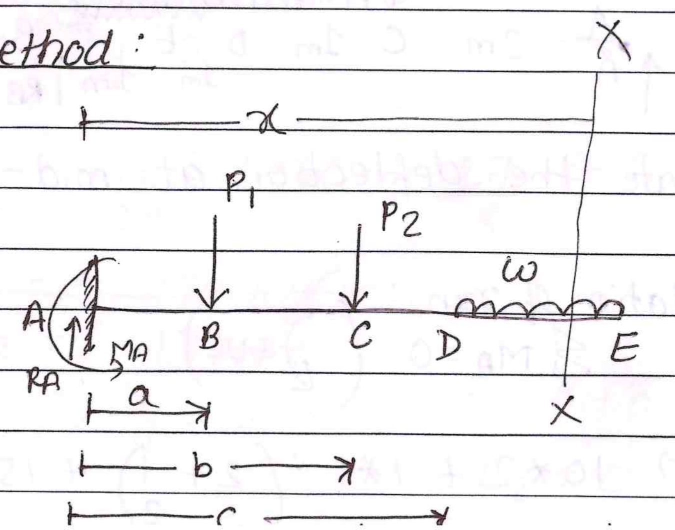
Deflection is the perpendicular distance between original axis and elastic curve.

Methods:

1. Double integration method
2. Moment area method
3. Conjugate beam method

Macaulay's method:

Illustration:



$$M_{x-x} = -M_a + R_a \cdot x - P_1(x-a) - P_2(x-b) - \frac{w(x-c)^2}{2}$$

So;

$$EI\frac{d^2y}{dx^2} = M$$

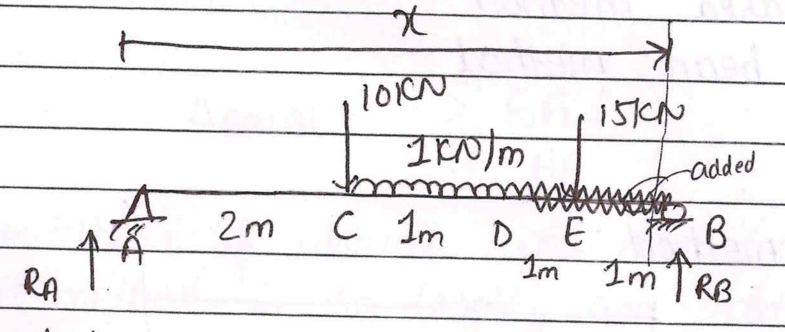
$$\Rightarrow EI\frac{d^2y}{dx^2} = -M_a + R_a \cdot x - P_1(x-a) - P_2(x-b) - \frac{w(x-c)^2}{2}$$

$$\Rightarrow EI\frac{dy}{dx} = -M_a x + \frac{R_a x^2}{2} - \left[\frac{P_1(x-a)^2}{2}\right] - \left[\frac{P_2(x-b)^2}{2}\right] - \left[\frac{w(x-c)^3}{6}\right] + C_1$$

$$\Rightarrow EI \cdot y = -\frac{M_a x^2}{2} + \frac{R_a x^3}{6} - \left[\frac{P_1(x-a)^3}{6}\right] - \left[\frac{P_2(x-b)^3}{6}\right] - \left[\frac{w(x-c)^4}{24}\right] + C_1 x + C_2$$

The term inside bracket indicates that if $x < a, b, c$; the whole term is neglected.

use boundary condition and get 'C_1 & C_2'

Q 

Calculate the deflection at mid-span.

Calculation of rxn:

$\sum M_A = 0$ (↻ +ve)

$$\Rightarrow 10*2 + 1*1*\left(2+\frac{1}{2}\right) + 15*4 - R_B*5 = 0$$

$$\Rightarrow R_B = 16.5 \text{ kN } (\uparrow).$$

$\sum F_y = 0$ (↑ +ve)

$$\Rightarrow R_A - 10 - 1*1 - 15 + 16.5 = 0$$

$$\Rightarrow R_A = 9.5 \text{ kN } (\uparrow).$$

$$M = R_A \cdot x - 10*(x-2) - 1*\frac{(x-2)^2}{2} - 15*(x-4) + 1*\frac{(x-3)^2}{2}$$

So;

$$EI\frac{d^2y}{dx^2} = M$$

$$\Rightarrow EI\frac{d^2y}{dx^2} = R_A \cdot x - 10(x-2) - \frac{(x-2)^2}{2} - 15(x-4) + \frac{(x-3)^2}{2}$$

Int.

$$EI\left(\frac{dy}{dx}\right) = \frac{R_A x^2}{2} - 10\frac{(x-2)^2}{2} - \frac{(x-2)^3}{6} - 15\frac{(x-4)^2}{2} + \frac{(x-3)^3}{6} + C_1$$

Int.

$$EI \cdot y = \frac{R_A x^3}{6} - 10\frac{(x-2)^3}{6} - \frac{(x-2)^4}{24} - 15\frac{(x-4)^3}{6} + \frac{(x-3)^4}{24} + C_1 x + C_2$$

Using boundary condition:

At $x = 0$; $y = 0$.

$$\therefore \frac{\cancel{270}\,80}{6} - \frac{16}{24} + 160 + \frac{27}{8} + C_2 = 0.$$

$$\Rightarrow \cancel{C_2 = -170.625}$$

$$\Rightarrow C_2 = -176.04$$

also at $x = 5$; $y = 0$. $\therefore 0 = 0 + 0 + 0 + 0 + 0 + C_2$

$\Rightarrow 0 =$

$$\Rightarrow C_2 = 0.$$

At $x=5$; $y=0$

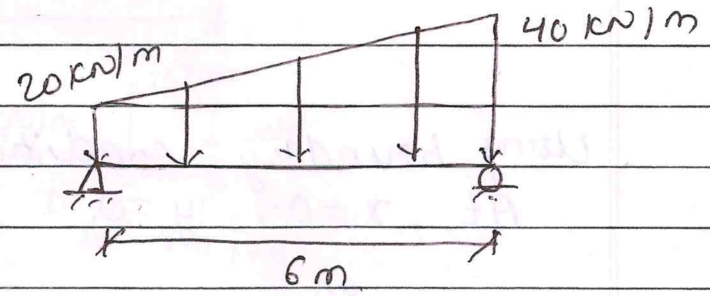
$$\therefore 0 = 9.5\frac{(5)^3}{6} - 10\frac{(5-2)^3}{6} - \frac{(5-2)^2}{4} - 15\frac{(5-4)^3}{6} + \frac{(5-3)^4}{24} + C_1 \cdot 5 + 0$$

$$\Rightarrow C_1 = \frac{893}{30}$$

$$\therefore EI \cdot y = 9.5\frac{x^3}{6} - 10\frac{(x-2)^3}{36} - \frac{(x-2)^4}{24} - 15\frac{(x-4)^3}{6} + \frac{(x-3)^4}{24} + \frac{893x}{30}$$

$\therefore$ At mid-span; $x = 2.5$. we neglect ?? ask

Q) A ~~grad~~ girder of span 6 m carries a distributed load varying from 20 kN/m at one end to 40 kN/m at another end. Calculate the position and magnitude of maximum deflection. Use double integration method.



Limitation of Double integration method:

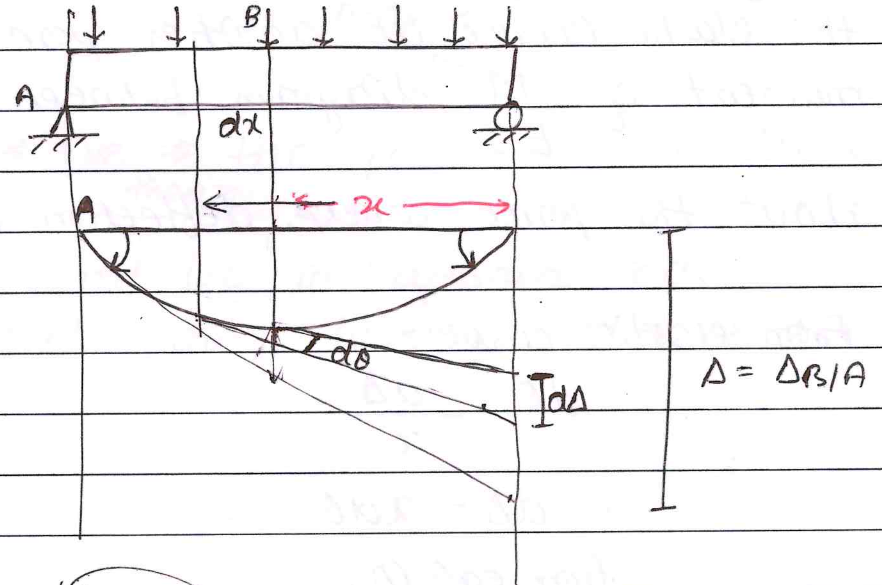
① It's not possible to use it when EI of the member vary with segment.

② Not used when loading is discontinious.

Moment area method:

Theorem-1:

The change in slope between 2 points in a straight member under flexure is equal to area of $\frac{M}{EI}$ diagram between these two points.



$\theta = \frac{l}{r}$

$(\because R = \frac{dx}{d\theta})$

$$\frac{M}{EI} = \frac{d\theta}{dx}$$

$$\Rightarrow d\theta = \frac{M}{EI}dx \quad \text{①}$$

$\Delta_{B/A}$ = Deflection at any point B in elastic curve measured from tangent through A.

we know; $\frac{M}{I} = \frac{\sigma}{y} = \frac{E}{R}$

If we require change in angle/slope between 'A' and 'B'; we should integrate eqn ① from 'A' to 'B'.

$$\int_A^B d\theta = \int_A^B \frac{M}{EI}dx$$

$$\Rightarrow \theta_B - \theta_A = \int_A^B \frac{M}{EI}dx = \text{area of } \frac{M}{EI} \text{ diagram}$$

between 'A' & 'B'.

Hence ; theorem is proved.

Theorem-2 :

Deflection at any point in the elastic curve of beam in the direction perpendicular to it's original straight line position measured from the tangent to the elastic curve at another point is given by moment of $\frac{M}{EI}$ diagram between the points and about the point where deflection is required.

From elastic curve ;

$$d\theta = \frac{d\Delta}{x}$$

$$\Rightarrow d\Delta = x\,d\theta$$

from eqⁿ ① ;

$$d\Delta = x \cdot \frac{M}{EI} dx \quad -③$$

If we require the distance between elastic curve at 'B' from tangent drawn through 'A'.

Then ;

$\Delta_{B/A}$ or Δ

$$\int_A^B d\Delta = \int_A^B \left(\frac{M}{EI} dx\right) x \quad -④$$

$$\Rightarrow \Delta_{B/A} = \int_A^B \frac{M}{EI} dx \cdot x = \text{Moment of } \frac{M}{EI} \text{ diagram}$$

between 'A' and 'B' about 'B'.

Sign Convention :

Sagging ⇒ +ve

Hogging ⇒ −ve

Slope ⇒ area ⇒ net area $\underset{\text{sagging}}{+ve}$ then anticlockwise

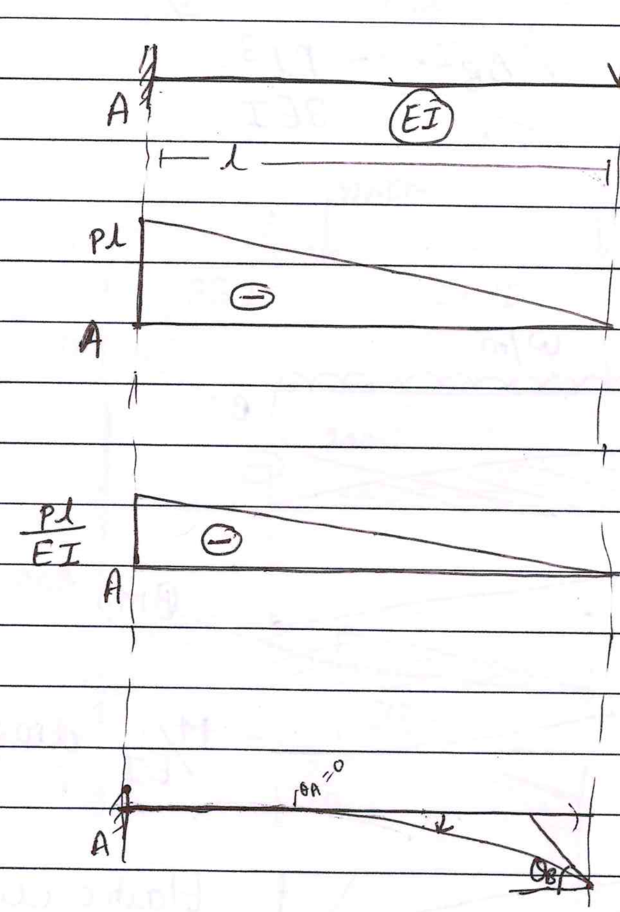
⇒ net area $\underset{\text{hogging}}{-ve}$ then clockwise.

Deflection ⇒ +ve $\underset{\text{then}}{\Rightarrow}$ the point where deflection is calculated lies in upward side w.r. to the tangent drawn through another point.

Q

$\Delta B = ?$

$\theta B = ?$



← B.M. diagram

$\frac{M}{EI}$ diagram

Elastic curve

so;

$\theta_B - \theta_A$ = area of $\frac{M}{EI}$ diagram betn 'A' & 'B'

Here; $\theta_A = 0$

$\theta_B = \frac{1}{2} * \left(\frac{-Pl}{EI}\right) * l$

$\Rightarrow \theta_B = \frac{-Pl^2}{2EI}$

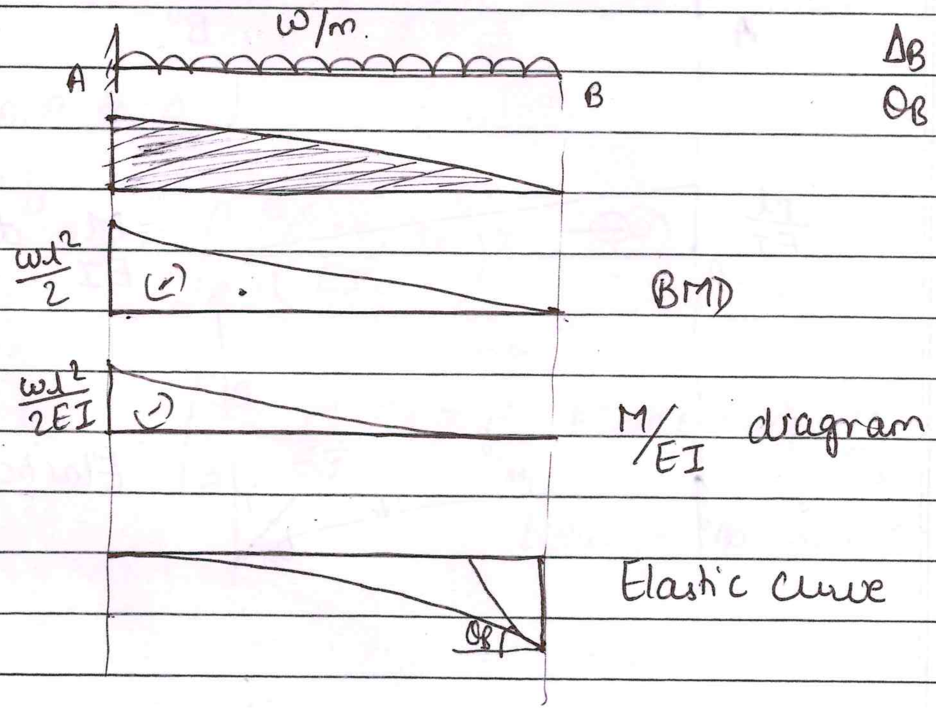
for deflection:

$\Delta_{B/A} = \Delta_B - \cancel{\Delta_A}^{0} = \Delta_B$ = moment of $\frac{M}{EI}$ diagram between 'A' & 'B' about 'B'.

$\Rightarrow \Delta_{B/A} = \Delta_B = \frac{1}{2} * \left(\frac{-Pl}{EI}\right) * l * \frac{2l}{3}$

~~$\Rightarrow \Delta_B$~~ $\Rightarrow \Delta_B = -\frac{Pl^3}{3EI}$

Q



$\Delta_B = ?$
$\theta_B = ?$

Here;

$\theta_B - \cancel{\theta_A}^{0}$ = area of $\frac{M}{EI}$ diagram betn A & B.

$\Rightarrow \theta_B = \frac{\cancel{1} * \left(\frac{-wl^2}{2EI}\right) * l}{3EI}$

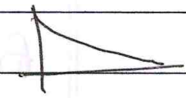
$\Rightarrow \theta_B = \frac{-wl^3}{6EI}$

also;

$\Delta_{B/A} = \Delta_B - \cancel{\Delta_A}^{0}$ = moment of $\frac{M}{EI}$ diagram between 'A' & 'B' about 'B'

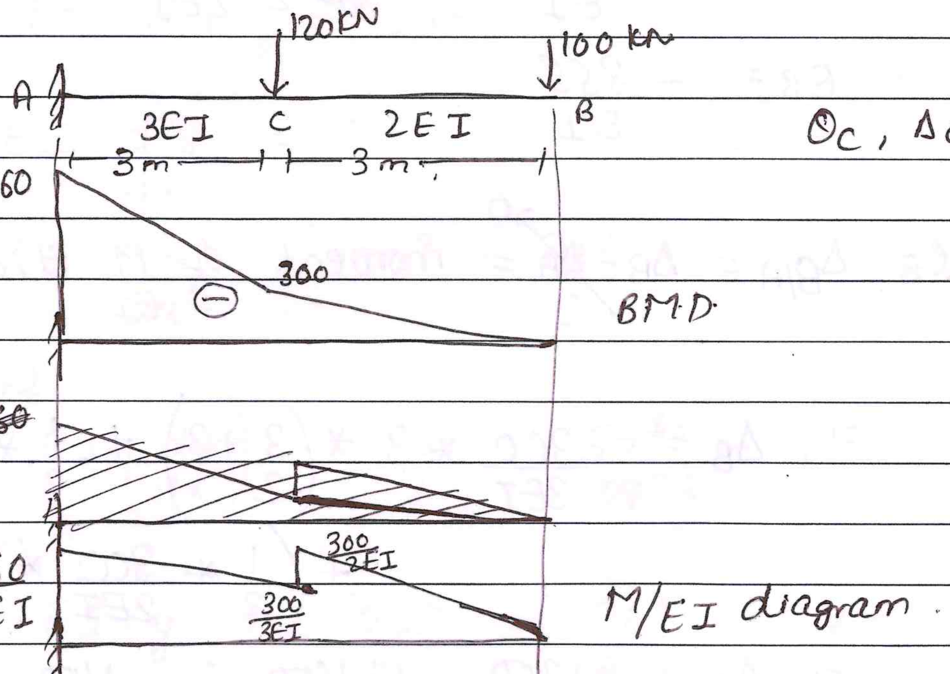
$\Rightarrow \Delta_{B/A} = \Delta_B = \frac{\left(\frac{-wl^2}{2}\right) * l * \frac{3l}{4}}{3}$

$\Rightarrow \Delta_{B/A} = -\frac{wl^4}{8EI}$

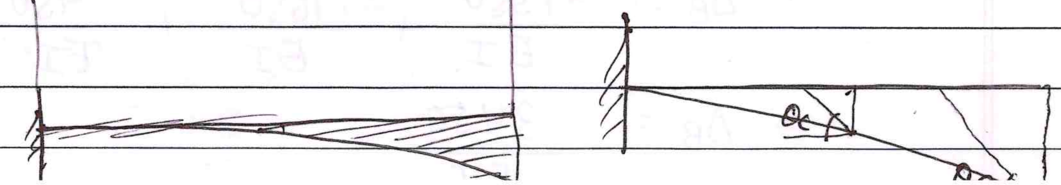


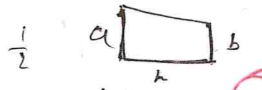
$\bar{x}, \bar{y} = \left(\frac{3a}{4}, \frac{a}{4}\right)$

Q



$\theta_C, \Delta_C, \Delta_B, \theta_B = ?$





$\frac{1}{2}(a+b)h$

$\theta_C - \cancel{\theta_A}^{0} = \frac{M}{EI}$ diagram area

$\Rightarrow \theta_C = \frac{1}{2} * \left(\frac{960}{3EI} - \frac{300}{3EI}\right) * 3 + \left(\frac{300}{3EI} * 3\right)$

$\Rightarrow \theta_C = \frac{630}{EI}$

also;

$\Delta_{C/A} = \Delta_C - \cancel{\Delta_A}^{0} =$ moment of $\frac{M}{EI}$ diagram

$\Rightarrow \Delta C = \left(\frac{300}{3EI} * 3\right) * \frac{3}{2} + \left(\frac{960}{3EI} - \frac{300}{3EI}\right) * 3 * \frac{2 \cdot 3}{3}$

$\Rightarrow \Delta C = -\frac{1110}{EI}$

for B:

$\theta_B - \cancel{\theta_A}^{0} = \frac{M}{EI}$ diagram area

$\Rightarrow \theta_B = \frac{630}{EI} + \frac{1}{2} * \frac{300}{2EI} * 3$

$\Rightarrow \theta_B = -\frac{855}{EI}$

$\Delta_{B/A} = \Delta_B - \cancel{\Delta_A}^{0} =$ moment of $\frac{M}{EI}$ diagram

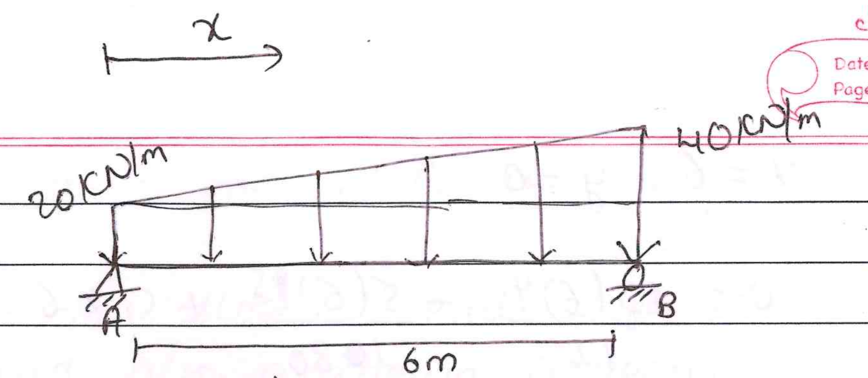
$\Rightarrow \Delta_B = \frac{300}{3EI} * 3 * \left(\frac{3}{2} + 3\right) + \frac{1}{2} * \left(\frac{960}{3EI} - \frac{300}{3EI}\right) * 3 * \left(3 + \frac{2 \cdot 3}{3}\right) + \left(\frac{1}{2} * \frac{300}{2EI} * 3\right) * \frac{2 \cdot 3}{3}$

from side B
can do from side 'A' as well

$\Rightarrow \Delta_B = \frac{1350}{EI} + \frac{1650}{EI} + \frac{450}{EI}$

$\Rightarrow \Delta_B = \frac{3450}{EI}$

Q.



$\sum M_A = 0$ (↻ +ve)

$\Rightarrow -R_B * 6 + 20 * 6 * \frac{6}{2} + \frac{1}{2} * 20 * 6 * \left(\frac{2}{3} * 6\right) = 0$

$\Rightarrow R_B = 100$ kN (↑)

also;

$R_A + R_B = 20 * 6 + \frac{1}{2} * 6 * 20$

$\Rightarrow R_A + 100 = 120 + 60$

$\Rightarrow R_A = 80$ kN (↑)

$\frac{20}{6} = \frac{?}{x}$

Now;

$M = 20 * x * \frac{x}{2} - \frac{1}{2} * x * \frac{10x}{3} * \frac{1}{3} . x$

$\Rightarrow M = 10x^2 - \frac{5x^3}{9}$

$\therefore EI \frac{d^2y}{dx^2} = M$

$\Rightarrow EI \frac{d^2y}{dx^2} = 10x^2 - \frac{5x^3}{9}$

Int.

$EI\left(\frac{dy}{dx}\right) = \frac{10x^3}{3} - \frac{5x^4}{36} + C_1$

Int.

$EI . y = \frac{10x^4}{12} - \frac{5x^5}{180} + C_1 x + C_2$

At $x = 0$; $y = 0$.

$\therefore C_2 = 0$

At $x=6$; $y=0$.

$$\therefore\ 0 = \frac{10(6)^4}{12} - \frac{5(6)^{\cancel{4}5}}{1080} + C_1 * 6$$

$$\cancel{\Rightarrow C_1 =} \Rightarrow 0 = 1080 - \overset{216}{\cancel{60}} + 6C_1$$

$$\Rightarrow C_1 = \cancel{-170}\ -864$$

$$\therefore\ EIy = \frac{10x^4}{12} - \frac{5x^{\cancel{4}5}}{1080} \cancel{-170x} - 864x$$

Diff.

$$EI\frac{dy}{dx} = \frac{40x^3}{12} - \frac{25x^4}{1080} \cancel{-170} - 864$$

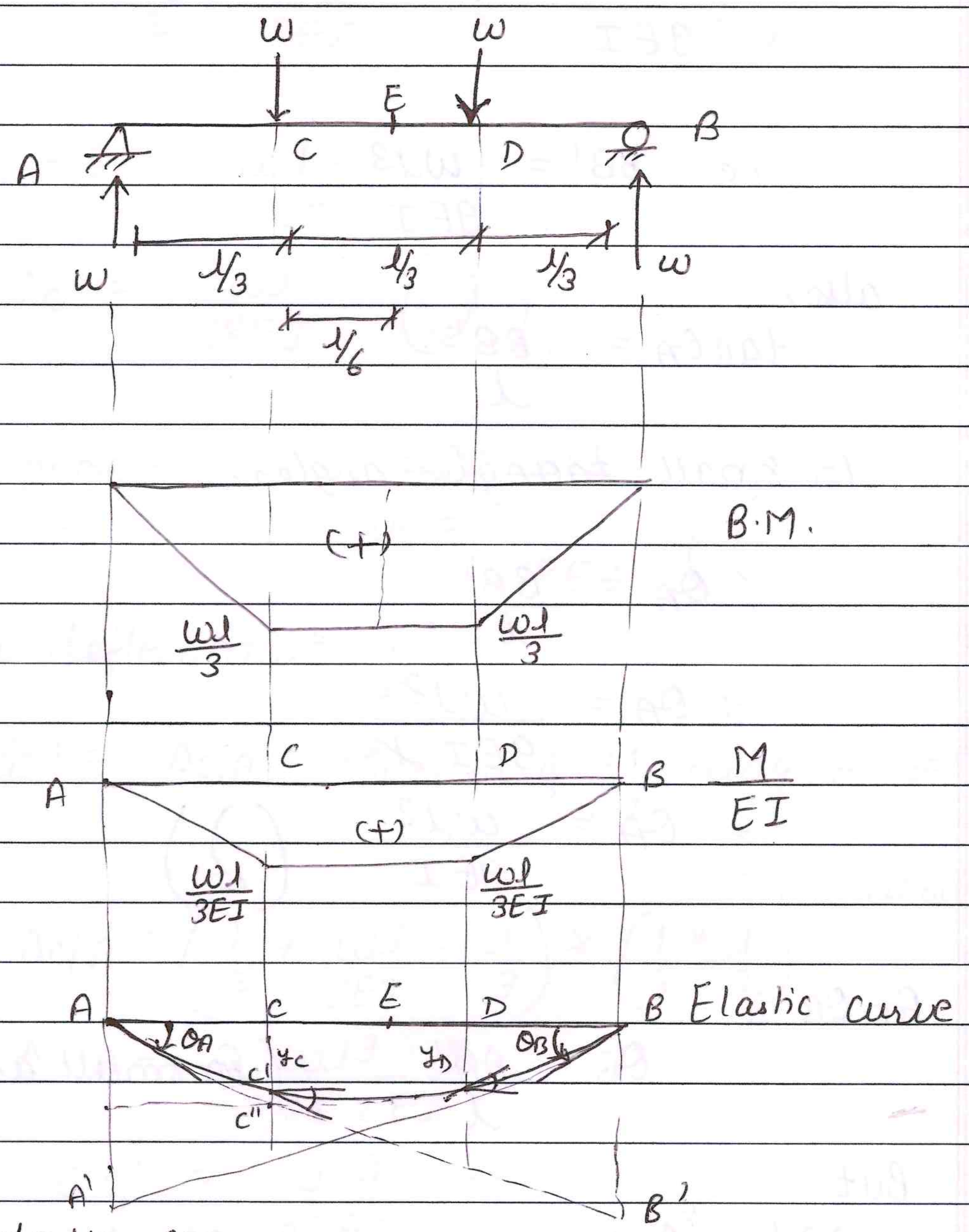
At max. defn; $\frac{dy}{dx} = 0$.

$$\Rightarrow 0 = \frac{40x^3}{12} - \frac{25x^4}{180} - 864$$

Solving;

Moment area method (Ctd) :-

Q Calculate the slope at ends, slope and deflection under loads and also maximum deflection.



From elastic curve;

$BB' = \Delta_{B/A} =$ Moment of $\frac{M}{EI}$ diagram between 'A' and 'B' about 'B'.

$$= \frac{1}{2} * \frac{Wl}{3EI} * \frac{l}{3} * \left(\frac{1}{3}\cdot\frac{l}{3} + \frac{2l}{3}\right) + \left(\frac{Wl}{3EI} * \frac{l}{3}\right) * \left(\frac{l}{6} + \frac{l}{3}\right) + \left(\frac{1}{2} * \frac{Wl}{3EI} * \frac{l}{3}\right) * \frac{2}{3}\cdot\frac{l}{3}$$

$$= \frac{wl^2}{18EI} * \frac{7l}{9} + \frac{wl^2}{9EI} * \frac{l}{2} + \frac{wl^2}{18EI} * \frac{2l}{9}$$

$$= \frac{wl^3}{18EI} + \frac{wl^3}{18EI}$$

$$= \frac{wl^3}{9EI}$$

i.e. $BB' = \frac{wl^3}{9EI}$

also;

$$\tan\theta_A = \frac{BB'}{l}$$

For small ~~triangles~~ angles;

$$\theta_A = \frac{BB'}{l}$$

$$\Rightarrow \theta_A = \frac{wl^{\cancel{3}2}}{9EI.\cancel{l}}$$

$$\Rightarrow \theta_A = \frac{wl^2}{9EI}$$

For θ_B:

$$\theta_B = \frac{AA'}{l} \quad \text{(for small angle)}$$

But

$AA' = \Delta A/B$ = moment of $\frac{M}{EI}$ diagram betn A & B about 'A'.

$$= \frac{wl^3}{9EI} \quad (\because \text{symmetric loading})$$

and $\theta_B = \frac{wl^2}{9EI}$

To get slope at 'c':

θ_{AC} = area of $\frac{M}{EI}$ diagram betn A & C,

$$\Rightarrow \theta_A - \theta_C = \frac{1}{2} * \frac{wl}{3EI} * \frac{l}{3} = \frac{wl^2}{18EI}$$

$$\Rightarrow \frac{wl^2}{9EI} - \theta_C = \frac{wl^2}{18EI}$$

$$\Rightarrow \theta_C = \frac{wl^2}{18EI}$$

As symmetric; $\theta_C = \theta_D$.

$$\therefore \theta_D = \frac{wl^2}{18EI}$$

To get deflection at 'c';

$C'C'' = \Delta C/A$ = Moment of $\frac{M}{EI}$ diagram betn A & C about C.

$$\Rightarrow \Delta C/A = \left(\frac{1}{2} * \frac{wl}{3EI} * \frac{l}{3}\right) * \left(\frac{1}{3} * \frac{l}{3}\right)$$

$$\Rightarrow \Delta C/A = \frac{wl^3}{162EI}$$

$$\Rightarrow C'C'' = \frac{wl^3}{162EI}$$

~~$\therefore y_C =$~~ From fig;

~~$y_C = CC''$~~ $\tan\theta_A = \frac{CC''}{AC}$

For small angles;

$$\theta_A = \frac{CC''}{AC}$$

$$\Rightarrow \frac{wl^2}{9EI} = \frac{CC''}{(l/3)}$$

$\Rightarrow CC'' = \frac{wl^3}{27EI}$

$\therefore y_C = CC' = CC'' - C'C''$

$\Rightarrow y_C = CC' = \frac{wl^3}{27EI} - \frac{wl^3}{162EI}$

$\Rightarrow y_C = \frac{5wl^3}{162EI}$

Due to symmetricity;

$$y_D = y_C = \frac{5wl^3}{162EI}$$

<u>For maximum deflection:</u>

Due to symmetry; maximum defⁿ occurs at mid-span

i.e. at $x = \frac{l}{2}$ i.e. at 'E'.

$\therefore y_{max} = \Delta_{E/A}$ = moment of $\frac{M}{EI}$ diagram betⁿ A & E about E.

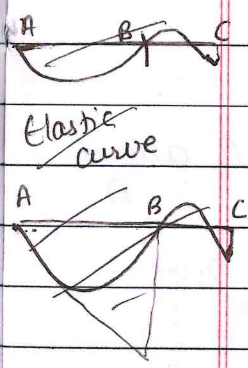
$= \cancel{\frac{1}{2}*}$

$= \cancel{\frac{1}{2}*\frac{wl}{3}*\frac{l}{3}*\frac{l}{3}+\frac{wl}{3}*\frac{l}{2}\frac{1}{3}}$

$= \left(\frac{1}{2}*\frac{wl}{3}*\frac{l}{3}\right)*\left(\frac{l}{6}+\frac{1}{3}\frac{l}{3}\right)+\left(\frac{wl}{3}*\frac{l}{6}\right)*\frac{1}{2}\cdot\frac{l}{6}$

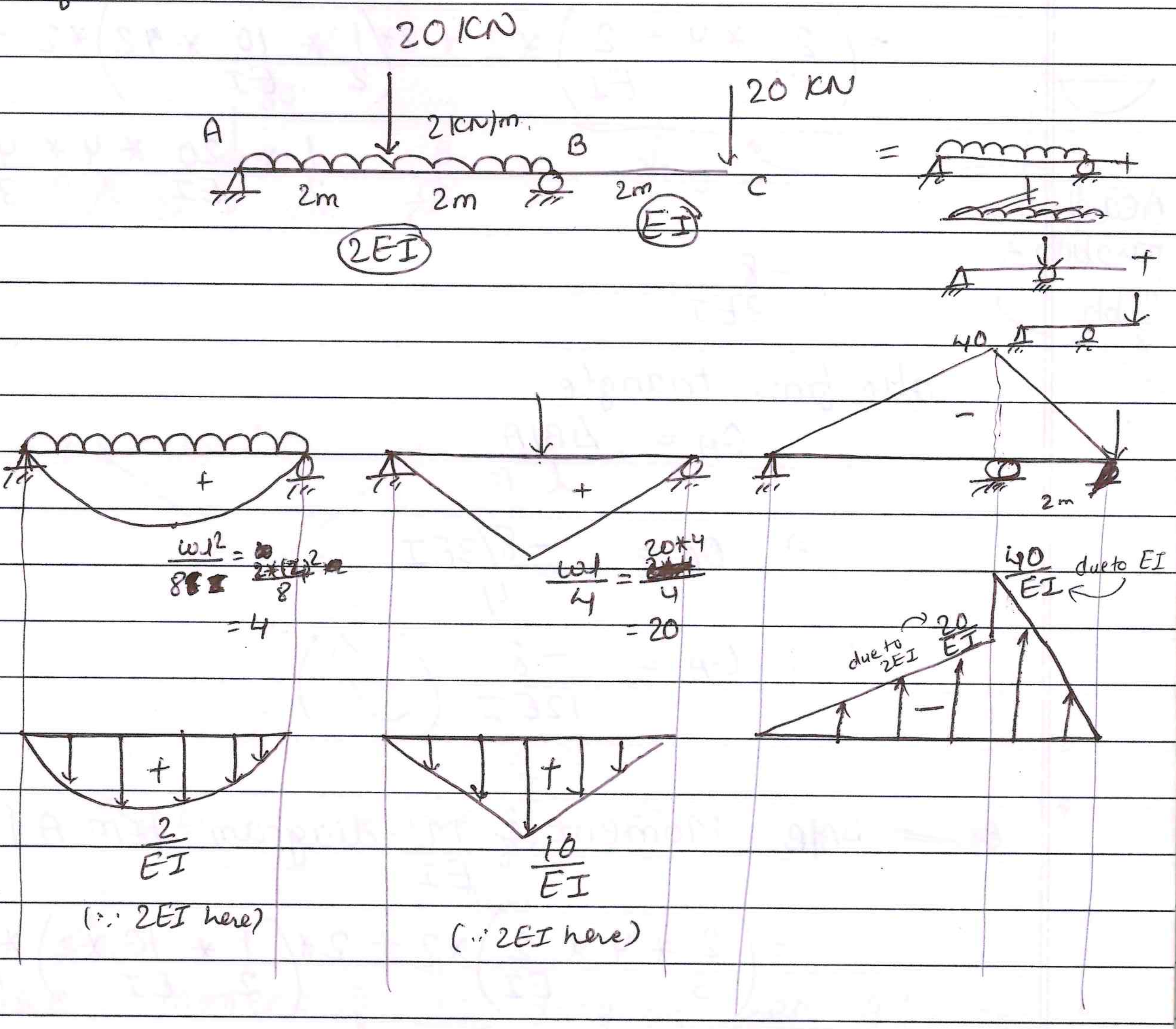
$= \frac{5wl^3}{486EI} + \frac{wl^3}{216EI}$

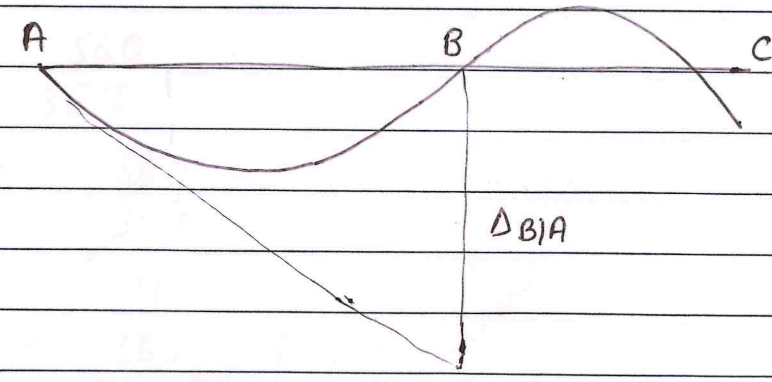
$= \frac{29wl^3}{1944EI}$

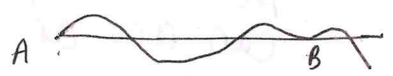


At overhanging condition; roller support can act as fixed support i.e. can restrict moment & HZ force as well.

Q) Calculate the slope at supports and slope & deflection at free ends.

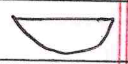






$\Delta_{B/A}$ = Moment of $\frac{M}{EI}$ diagram betn A & B about B

$$= \left(\frac{2}{3}*4*\frac{2}{EI}\right)*2 + 2*\left(\frac{1}{2}*\frac{10}{EI}*2\right)*2 - \frac{1}{2}*\frac{20}{EI}*4*\frac{4}{3}$$



Area of parabola = $\frac{2}{3}bh$

$$= \frac{-8}{3EI}$$

also from triangle;

$$\theta_A = \frac{\Delta_{B/A}}{l}$$

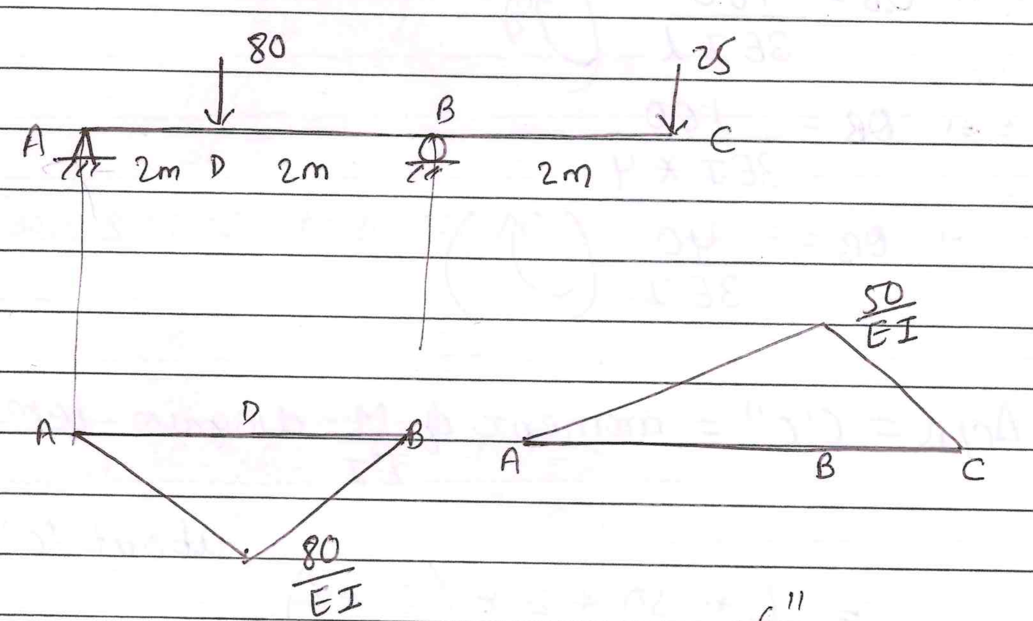
$$\Rightarrow \theta_A = \frac{-8/3EI}{4}$$

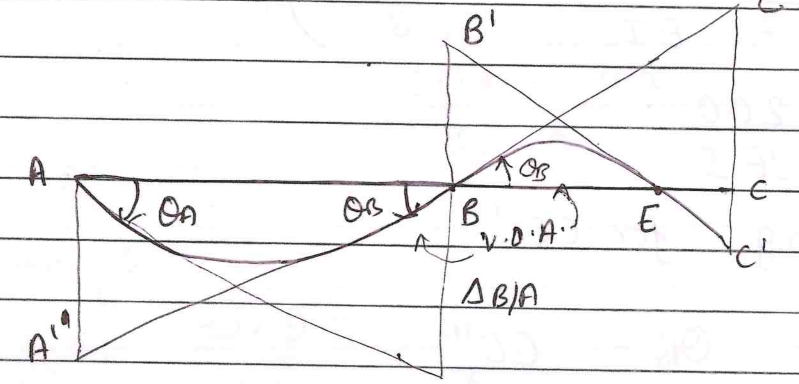
$$\Rightarrow \theta_A = \frac{-8}{12EI}$$

~~θB~~ $\Delta_{A/B}$ = Moment of $\frac{M}{EI}$ diagram betn A & B about A.

$$= \left(\frac{2}{3}*4*\frac{2}{EI}\right)*2 + 2*\left(\frac{1}{2}*\frac{10}{EI}*2\right)*2 - \frac{1}{2}*\frac{20}{EI}*4*\frac{4}{3} -$$

Q. Calculate slope at supports and slope and deflection at free end.





$\Delta_{B/A}$ = moment of $\frac{M}{EI}$ diagram betn A & B abad B.

$$= 2*\left(\frac{1}{2}*\frac{80}{EI}*2\right)*2 - \frac{1}{2}*\frac{50}{EI}*4*\frac{4}{3}$$

$1\frac{1}{3}\cdot 4$

$$= \frac{560}{3EI}$$

$$\theta_A = \frac{\frac{560}{3EI}}{4}$$

$$\Rightarrow \theta_A = \frac{560}{12EI}$$

$$\theta_B = \frac{\Delta_{A/B}}{l} = \frac{\text{Moment of } M/EI \text{ diagram betn A \& B about 'A'}}{l}$$

$$\Rightarrow l \cdot \theta_B = 2 * \left(\frac{1}{2} * \frac{80}{EI} * 24\right) * 2 - \frac{1}{2} * \frac{50}{EI} * 4 * \left(\frac{2 * 4}{3}\right)$$

$$\Rightarrow \theta_B = \frac{160}{3EI\,l}$$

$$\Rightarrow \theta_B = \frac{160}{3EI * 4}$$

$$\Rightarrow \theta_B = \frac{40}{3EI}$$

$\Delta_{C/B} = C'C'' =$ moment of $\frac{M}{EI}$ diagram betn 'B' & 'C' about 'C'.

$$= \frac{1}{2} * \frac{50}{EI} * 2 * \left(\frac{2}{3} * 2\right)$$

$$= \frac{200}{3EI}$$

From fig; for CC'';

$$\theta_B = \frac{CC''}{BC}$$

$$\Rightarrow CC'' = BC * \theta_B$$

$$\Rightarrow CC'' = \frac{\cancel{160}\ 40}{3EI} * 2 \quad (\because V.O.A.)$$

$$\Rightarrow CC'' = \frac{80}{3EI}$$

$$\therefore CC' = C'C'' - CC''$$

$$\Rightarrow CC' = \frac{200}{3EI} - \frac{80}{3EI}$$

$$\Rightarrow CC' = \frac{40}{EI}$$

$BB' =$ Moment of $\frac{M}{EI}$ diagram betn 'B' & 'C' about 'B'

$$= \left(\frac{1}{2} * \frac{50}{EI} * 2\right) * \frac{2}{3}$$

$$= \frac{100}{3EI}$$

From similar triangles BB'E & CC'E;

$$\frac{BB'}{BE} = \frac{CC'}{CE}$$

$$\Rightarrow \frac{100/3EI}{x} = \frac{40/EI}{2-x}$$

$$\Rightarrow \frac{100}{3x} = \frac{40}{2-x}$$

$$\Rightarrow 200 - 100x = 120x$$

$$\Rightarrow x = 0.91 \text{ m}$$

From Δ ECC';

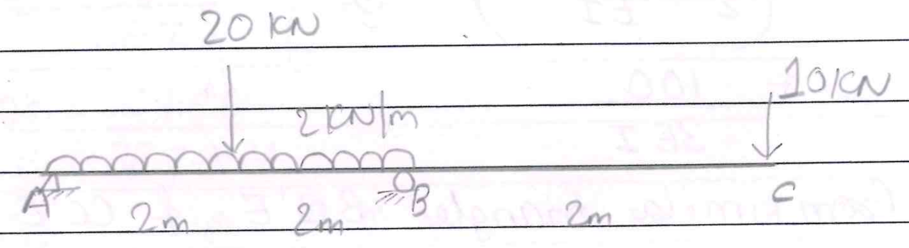
$$\tan\theta = \frac{CC'}{CE}$$

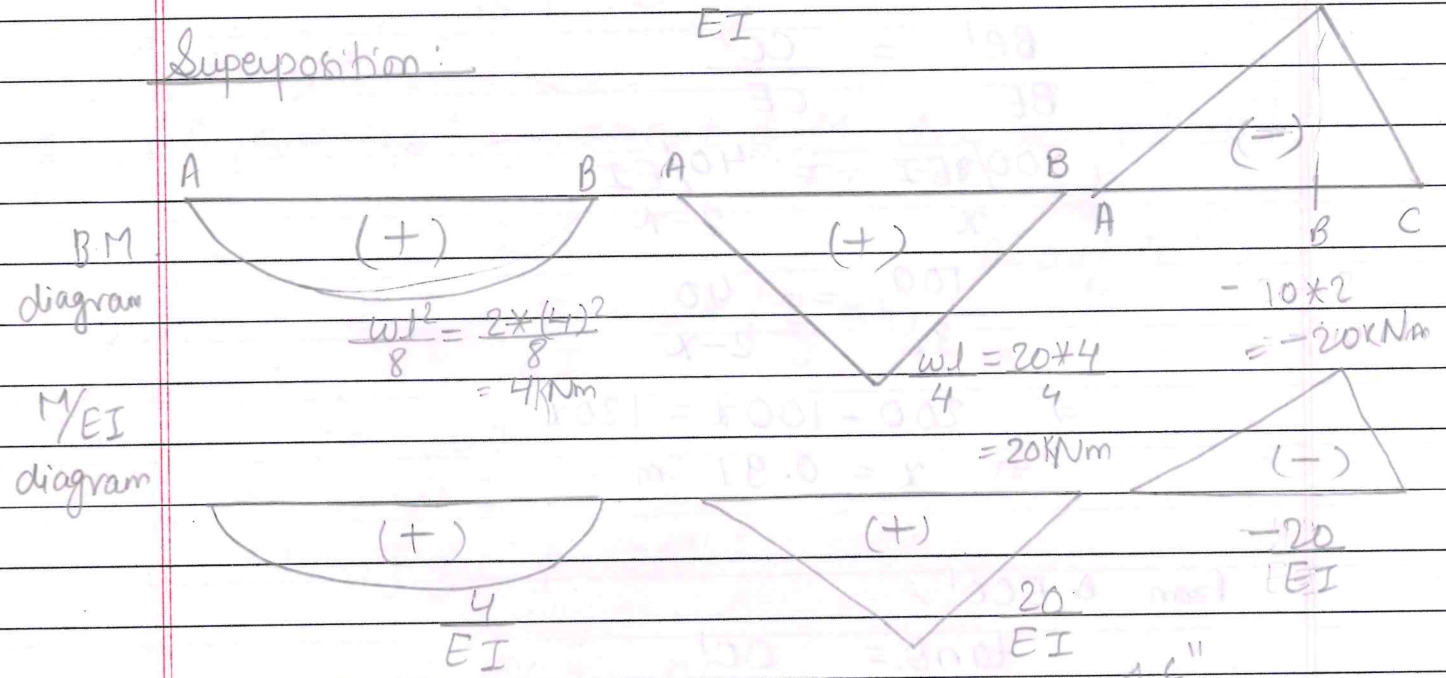
$$\Rightarrow \theta_C = \frac{CC'}{CE}$$

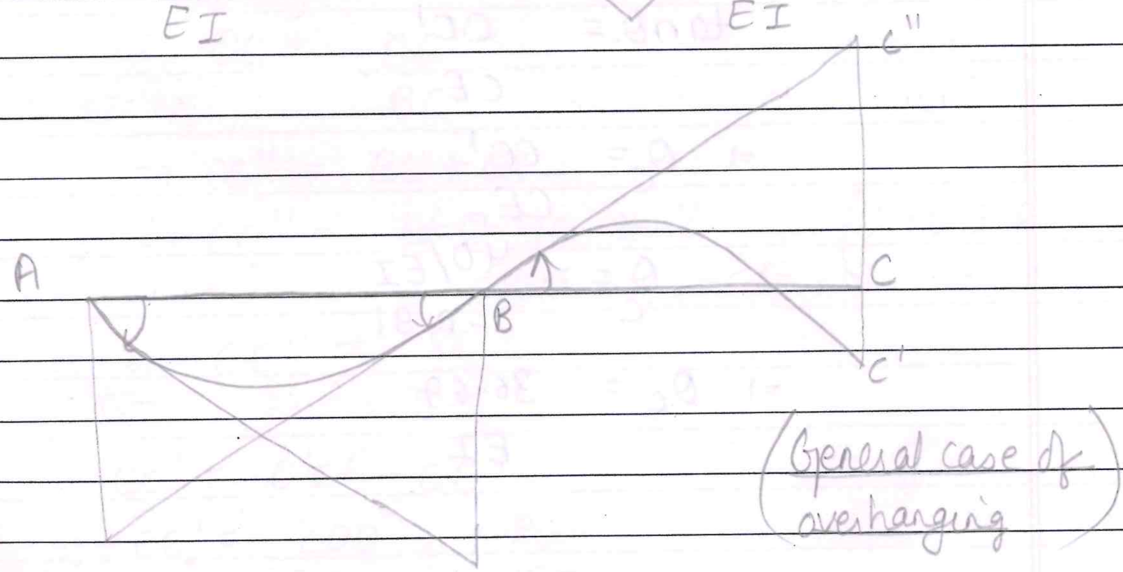
$$\Rightarrow \theta_C = \frac{40/EI}{2-0.91}$$

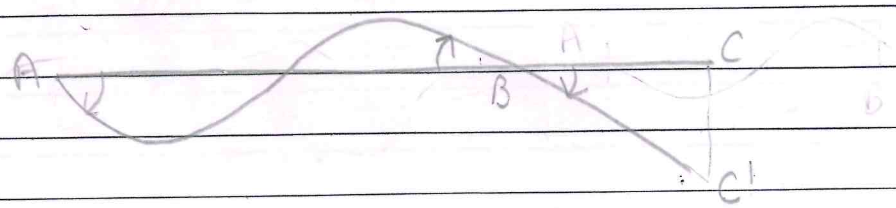
$$\Rightarrow \theta_C = \frac{36.67}{EI}$$

Q. Calculate the slope at supports and deflection at free ends









$$\Delta_{B/A} = \Delta_B - \cancel{\Delta_A}^{\,0} = \text{Moment of } \frac{M}{EI} \text{ diagram between AB}$$

about 'B'

$$\Rightarrow \Delta_{B/A} = \Delta_B = \underbrace{\left(\frac{2}{3} * 4 * \frac{4}{EI}\right)}_{} * 2 + 2*\left(\frac{1}{2} * 2 * \frac{20}{EI}\right)*2 - \left(\frac{1}{2} * 4 * \frac{20}{EI}\right) * \frac{1}{3}.4$$

Area of parabola $= \frac{2}{3}bh$

$$= \frac{48}{EI}$$

$\frac{64}{3} + 80 - 80 - \frac{320}{3}$

also from triangle;

$$\theta_A = \frac{\Delta_{B/A}}{l} \quad \text{(For small angles)}$$

$$\Rightarrow \theta_A = \frac{48/EI}{4}$$

$$\Rightarrow \theta_A = \frac{12}{EI}$$

$$\Delta_{A/B} = \Delta_A - \cancel{\Delta_B}^{\,0} = \text{Moment of } \frac{M}{EI} \text{ diagram bet}^n \text{ A \& B}$$

about 'A'

$$\Rightarrow \Delta_{A/B} = \Delta_A = \left(\frac{2}{3} * 4 * \frac{4}{EI}\right) * 2 + 2*\left(\frac{1}{2} * 2 * \frac{20}{EI}\right)*2 - \left(\frac{1}{2} * 4 * \frac{20}{EI}\right) * \frac{2}{3}.4$$

$$= \frac{64}{3EI} + \frac{80}{EI} - \frac{320}{3EI}$$

$$= \frac{-16}{3EI}$$

and $\theta_B = \frac{\Delta_{A/B}}{l} = \frac{-16}{3EI*4} = \frac{-4}{3EI}$

From triangle BCC'';

$$\tan\theta_B = \frac{CC''}{BC} \qquad (\theta_B \text{ as V.O.A})$$

For small angles;

$$\theta_B = \frac{CC'}{BC}$$

$$\Rightarrow \frac{-4}{3EI} = \frac{CC'}{2}$$

$$\Rightarrow CC' = \frac{-8}{3EI}$$

Limitations: Always elastic curve req^d to solve which is difficult in Complex loading

∴ Relative slope and def^n only obtained. Not exact obtained

Conjugate Beam method:

This method is a modified form of moment area method. In moment area method, we donot obtain the direct slope and deflection because the theorems are developed for change in slope and relative deflection. So; in case of complicated load it is difficult to track the elastic curve.

Hence: conjugate beam method are preferred in such cases which provides the solution faster and in convenient way.

Conjugate beam is a fictitious beam which shows the kinematic behaviour of real beam such that shear force at any point on conjugate beam is equal to slope in real beam and bending moment at any point on conjugate beam is equal to deflection at same point on real beam.

Theorem of conjugate beam:

When the conjugate beam is loaded by $\frac{M}{EI}$ diagram of real beam; then shear force at any point on conjugate beam is equal to ~~shear~~ slope at same point on real beam and B.M. at any point on conjugate beam is equal to deflection at same point on real beam.

Sign convention:

+ve shear i.e, ↑ ↓ = Clockwise slope
(Left ↑ Right ↓)

+ve B.M i.e. Sagging B.M. = downward deflection.

Characteristics of conjugate beam:

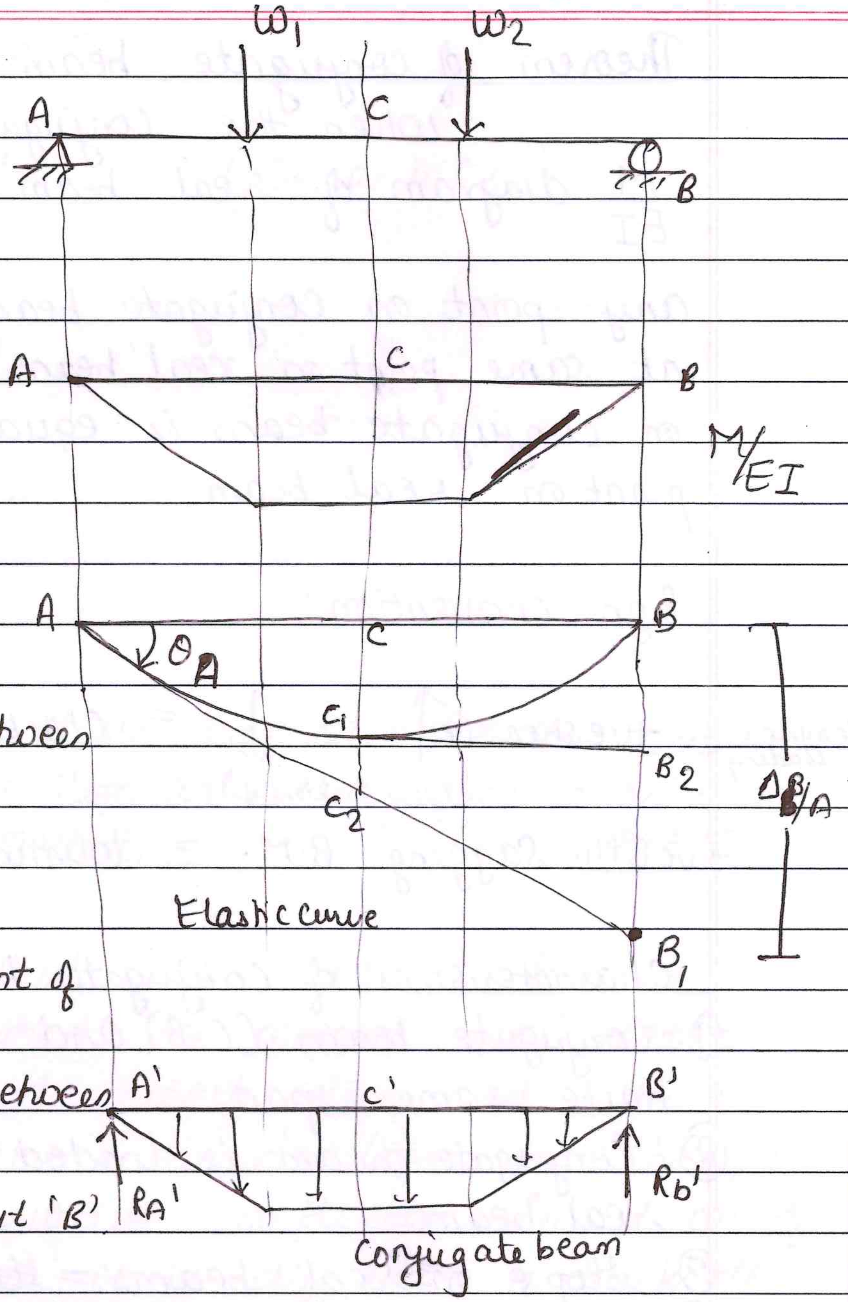
1. Conjugate beam (C.B) and real beam (R.B) both have same span.
2. Conjugate beam is loaded with $\frac{M}{EI}$ diagram of real beam.
3. Slope on real beam = ~~sto~~ S.F. on conjugate beam
4. Deflection on real beam = B.M. ~~on~~ on conjugate beam.

Illustration:

Let us consider a simply supported beam of length 'l' and subjected to two point loads 'w_1' and 'w_2' at distance l_1 and l_2 from 'A' as shown in figure.

~~Let~~

Let 'C' be the point at 'x' distance from 'A'.



From elastic curve;

$$\theta_A = \frac{BB_1}{l}$$

BB_1 = moment of $\frac{M}{EI}$ diagram between A & B about B.

$\Rightarrow \theta_A = \frac{1}{l} *$ moment of $\frac{M}{EI}$ diagram between 'A' & 'B' about 'B'.

Also;

Change in slope $(\theta_{A/C}) = \theta_A - \theta_C$
= area of $\frac{M}{EI}$ diagram between 'A' & 'C'.

$\therefore \theta_C = \theta_A -$ area of $\frac{M}{EI}$ diagram betn A & C.

$\Rightarrow \theta_C = \frac{1}{l} * \{$ Moment of $\frac{M}{EI}$ diagram betn A & B about B $\}$ $-$ $\{$area of $\frac{M}{EI}$ diagram betn A & C$\}$ — (1)

Similarly; to get deflection:

$$\theta_A = \frac{CC_2}{AC}$$

$$\Rightarrow CC_2 = x * \theta_A$$

C_1C_2 = Moment of $\frac{M}{EI}$ diagram between 'A' & 'C' about 'C'.

Deflection at C, $\Delta C = CC_2 - C_1C_2$

$\Delta C = x * \theta_A -$ moment of $\frac{M}{EI}$ diagram betn A & C about C.

$\therefore \Delta C = \frac{x}{l} * \{$Moment of $\frac{M}{EI}$ diagram betn A & B about B$\}$ $-$ $\{$Moment of $\frac{M}{EI}$ diagram betn A & C about C$\}$ — (2)

In fourth figure (conjugate beam):

$$\sum M_B' = 0.$$

$\Rightarrow R_a' * l -$ moment due to load at B $= 0$

$\Rightarrow R_a' = \frac{\text{moment due to load at B}}{l}$

$\Rightarrow R_a' = \frac{\text{moment of } \frac{M}{EI} \text{ diagram betn A \& B about B}}{l}$ — (3)

S.F. at 'C' ⇒

⇒ = Ra' − area of $\frac{M}{EI}$ diagram between A' & C'

⇒ S.F. at C' = $\frac{1}{l}$ * {moment of $\frac{M}{EI}$ diagram betn A & B about B} − {area of $\frac{M}{EI}$ diagram betn A' and C'} — (4)

Equating (1) & (4);

$$\boxed{\theta_C = S.F.\ at\ C'}$$

Also;

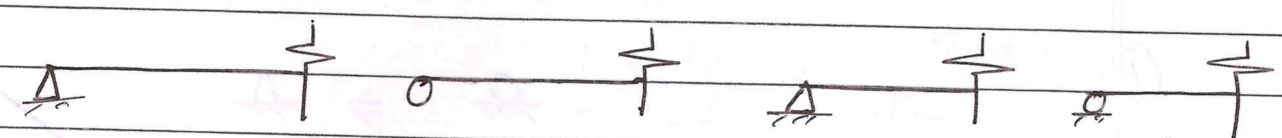
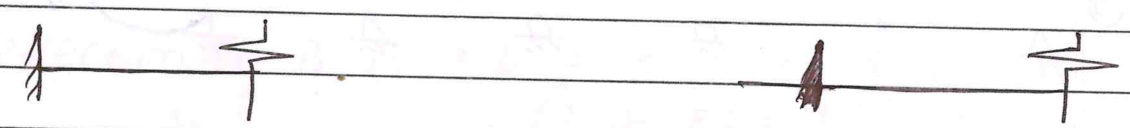
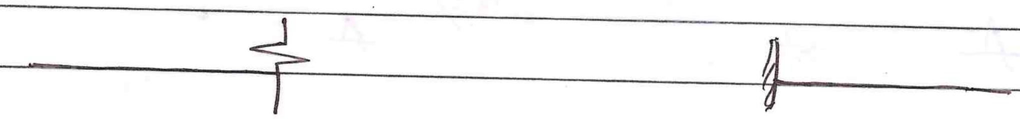
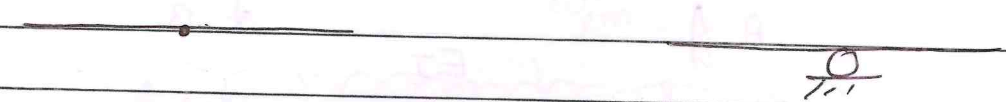
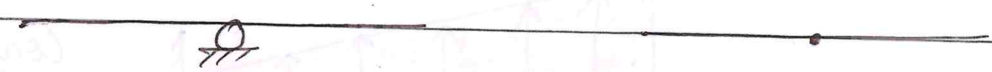
B.M. at C' = $Ra' * x$ − moment of $\frac{M}{EI}$ diagram betn A' & C' about C'.

→ B.M. at C' = $\frac{x}{l}$ * {moment of $\frac{M}{EI}$ diagram betn A & B about B} − {Moment of $\frac{M}{EI}$ diagram betn A' & C' about C'}

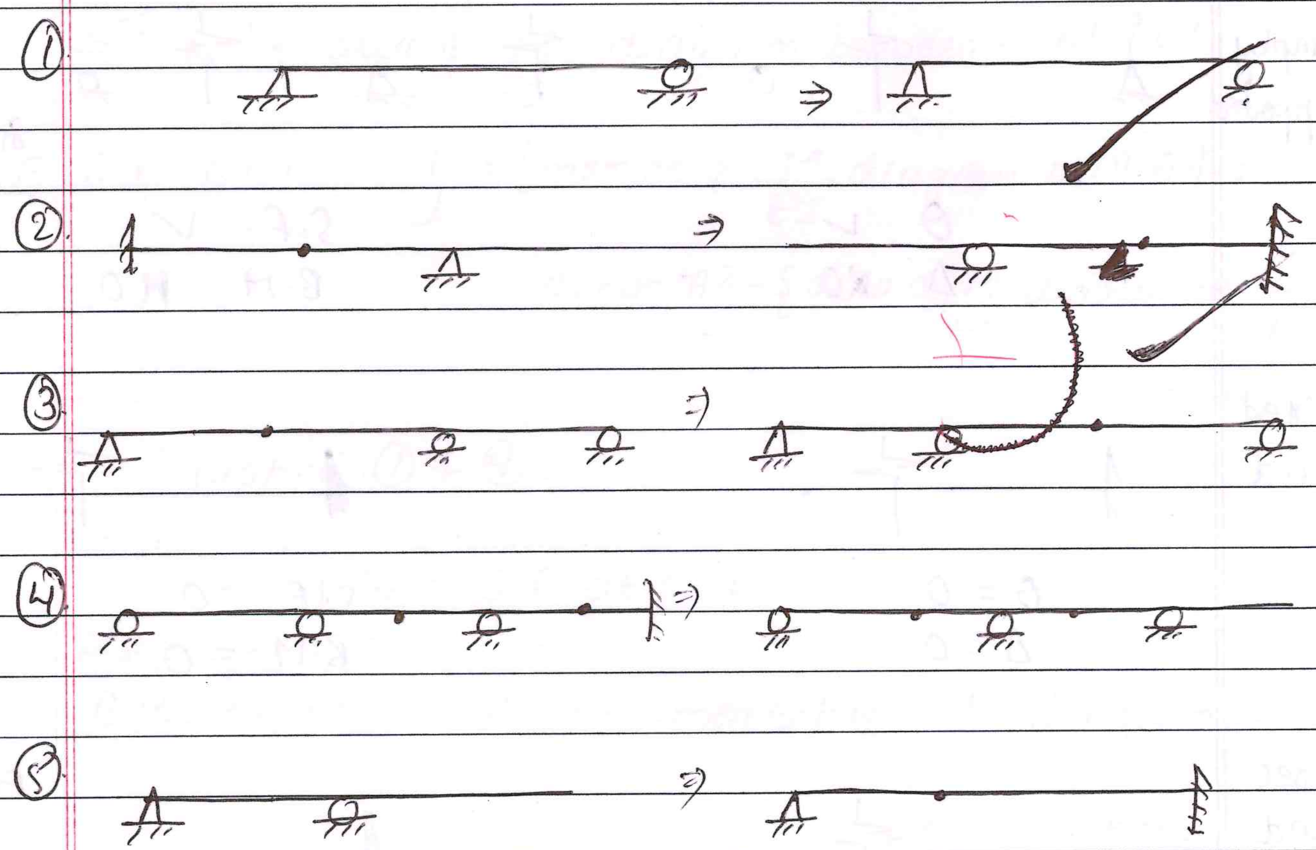
Equating (2) & (6);

$$\boxed{\Delta C = B.M.\ at\ C'}$$ ∴ proved

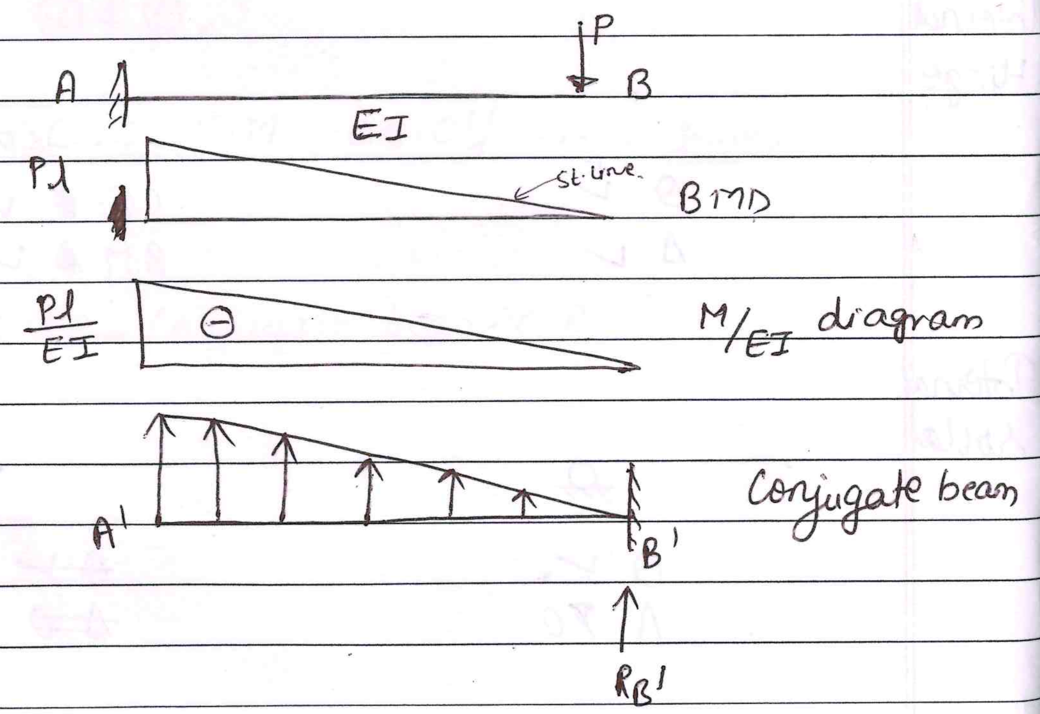
How to make conjugate beam (C.B.)?

	R.B.	C.B.
Simply supported	 θ ✓, Δ ~~✗~~ 0	S.F. ✓, B.M. 0 — Simply supported
Fixed end	 θ = 0, Δ = 0	S.F. = 0, B.M. = 0. — Free end
Free end	 θ ✓, Δ ✓	S.F. ✓, B.M. ✓ — Fixed end
Internal Hinge	 θ ✓, Δ ✓	S.F. ✓, B.M. ✓ — Internal roller.
Internal roller.	 θ ✓, Δ 0	S.F. ✓, B.M. ✗ — Internal roller.

~~Conju~~ Convert the following into conjugate beam:



Q. Calculate slope and deflection at free end of cantilever beam as shown in figure.



Here;

Slope at B in R.B. = S.F. at B' in C.B.

$$= \frac{1}{2} * \frac{Pl}{EI} * l \quad \cancel{\text{area}}$$

$$= \frac{Pl^2}{2EI} \quad (\text{+ve shear})$$

$$\therefore \theta_A = \frac{Pl^2}{2EI} \; (\downarrow).$$

real

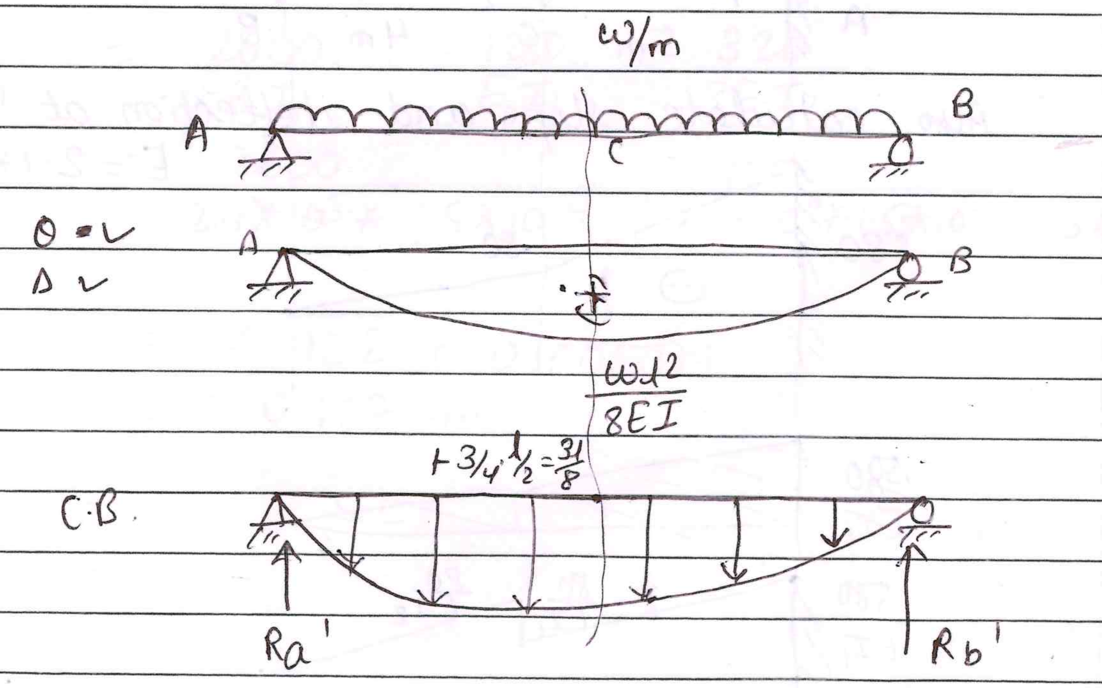
Deflection at B in R.B. = B.M. at B' in C.B.

$$= \left(\frac{1}{2} * \frac{Pl}{EI} * l\right) * \left(\frac{2}{3} \cdot l\right)$$

$$= \frac{Pl^3}{3EI} \quad (\text{sagging})$$

$$\therefore \Delta_B = \frac{Pl^3}{3EI} \; (\downarrow).$$

Q. Calculate the slope at ends and deflection at mid-span.



area of parabola

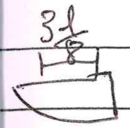
So; $R_a' = R_b' = \frac{1}{2} * \left(\frac{2}{3} * \frac{wl^2}{8EI} * l\right)$

$= \frac{wl^3}{24EI}$ (↑)

So;

Slope at 'A' (θ_A) = S.F. at 'A' = $R_a' = \frac{wl^3}{24EI}$

θ_B = S.F. at B' = $R_b' = \frac{-wl^3}{24EI}$.



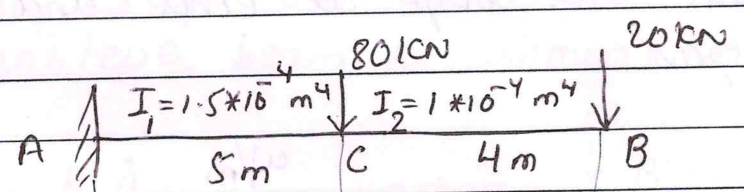
Deflection at mid-span = BM at mid-span of C.B.

$= \frac{wl^3}{24EI} * \frac{l}{2} - \left(\frac{2}{3} * \frac{l}{2} * \frac{wl^2}{8EI}\right) * \frac{3}{8} \cdot \frac{l}{2}$

area of parabola semi

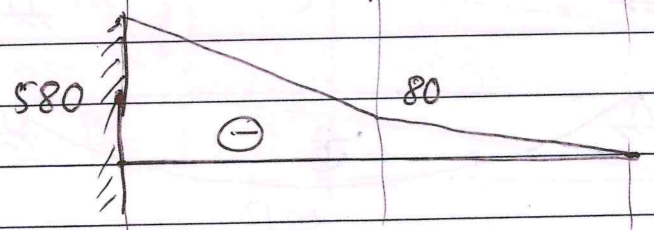
$= \frac{5wl^4}{384EI}$

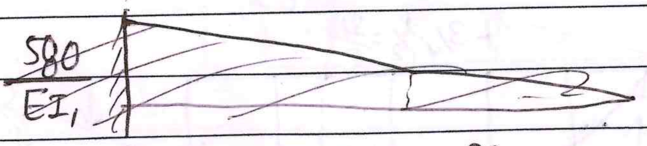
Q. Calculate the slope and deflection at free end of cantilever beam as shown in figure.

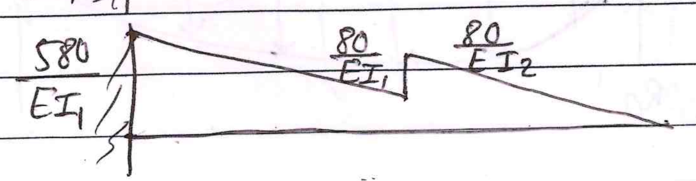


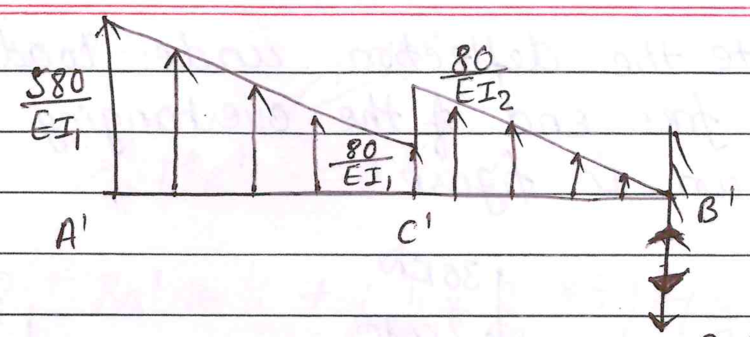
Also calculate slope and deflection at 'c'.

$E = 2.1*10^8\ kN/m^2$









C.B.

In C.B;

θ_B = S.F. at B'

$= R_B'$

$= +\left(\frac{160}{EI_2} + \frac{1650}{EI_1}\right)$

$= +\left(\frac{160*1000}{E*10^{-4}} + \frac{1650*1000}{E*1.5*10^{-4}}\right)$

~~$= 6*10^{-5}$~~

$= 0.06^c$

$R_B' = \frac{1}{2} * \frac{80}{EI_2} * 4 + \frac{1}{2}\left(\frac{580}{EI_1} + \frac{80}{EI_1}\right) * 5$

~~$= 1810$~~

$= \frac{160}{EI_2} + \frac{1650}{EI_1}$

Δ_B = B.M. at B'

~~$= \left(\frac{80}{EI_1} * 5\right) * \left(4 + \frac{5}{2}\right) + \frac{1}{2}\left(\frac{580}{EI_1} - \frac{80}{EI_1}\right) * 5 * \left(4 + \frac{2}{3} \cdot 5\right) + \frac{1}{2} * \frac{80}{EI_2} *$~~

$= \left(\frac{80}{EI_1} * 5\right) * \left(4 + \frac{5}{2}\right) + \frac{1}{2} * \left(\frac{580}{EI_1} - \frac{80}{EI_1}\right) * 5 + \frac{1}{2} * \frac{80}{EI_2} * \left(\frac{2}{3} * 4\right)$

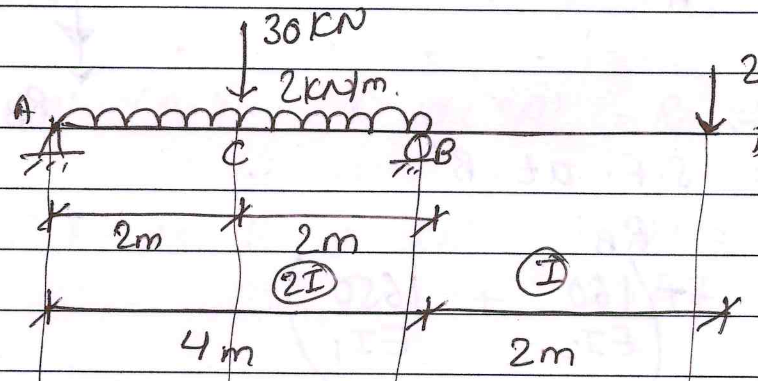
$= \frac{2600}{EI_1} + \frac{1250}{EI_1} + \frac{320}{3EI_2}$

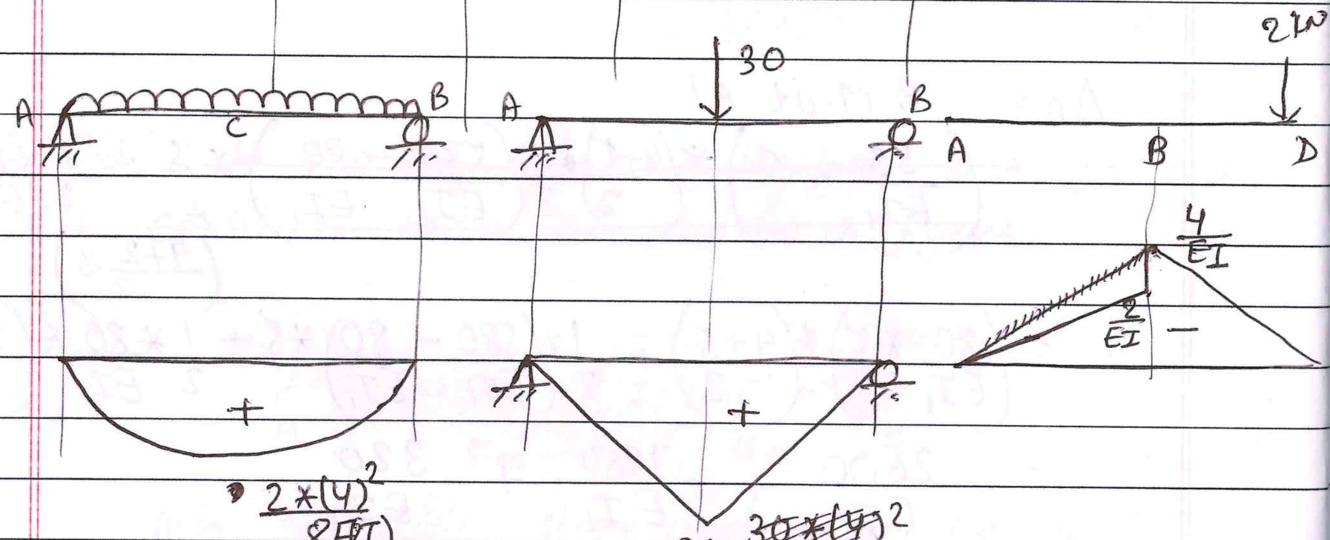
$= \frac{2600}{2.1*10^8 * 1.5*10^{-4}} + \frac{1250}{2.1*10^8*1.5*10^{-4}} + \frac{320}{3*2.1*10^8*10^{-4}}$

$= 0.122 + 0.005079$

$= 0.127$ m

Q) Calculate the deflection under load at 'c' and at free end of the overhanging beam loaded as shown in figure.

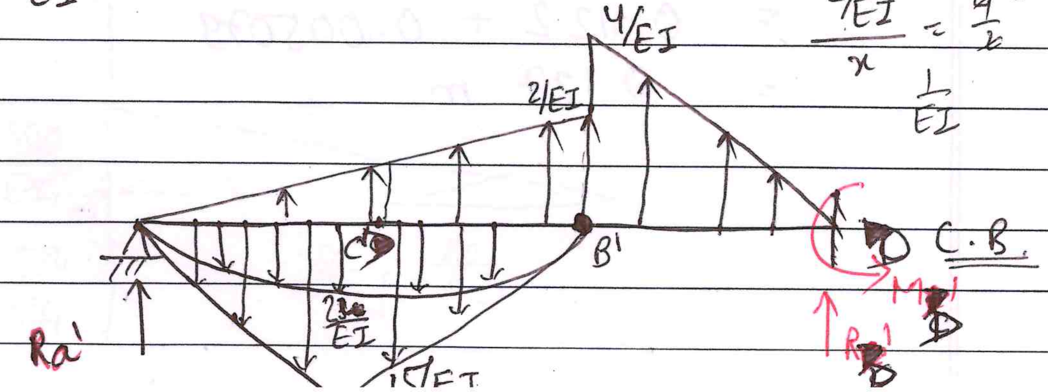




$\frac{2*(4)^2}{8EI} = \frac{4}{2EI} = \frac{2}{EI}$

$\frac{30}{2EI} = \frac{15}{EI}$ $\quad \frac{30*(4)^2}{8}$

$\frac{2/EI}{x} = \frac{4}{2}$ $\quad \frac{1}{EI}$



for reaction;

Hinge

$\sum M_{B'} = 0$

$\Rightarrow Ra' * 4 + \left(\frac{1}{2} * \frac{2}{EI} * 4\right) * \left(\frac{2}{3} * 4\right) - 2*\left(\frac{1}{2} * \frac{15}{EI} * 2\right) * 2 - \left(\frac{2}{3} * \frac{2}{EI} * 4\right) * 2 = 0$

$\Rightarrow Ra' = \frac{49}{3EI}$

also, $\sum F_y = 0$ (↑+ve)

$\Rightarrow \frac{49}{3EI} + \frac{2}{3} * \frac{1}{2} * \frac{2}{EI} * 4 - 2*\left(\frac{1}{2} * \frac{15}{EI} * 2\right) - \frac{2}{3} * \frac{2}{EI} * 4 - \frac{1}{2} * \frac{4}{EI} * 2 + R_B' = 0$

$\Rightarrow R_B' = \frac{+11}{EI}$ i.e. $\frac{11}{EI}$ (↓).

Now;

Δ_C = B.M. at C' in C.B.

$= \frac{49}{3EI} * 2 - \left(\frac{1}{2} * \frac{15}{EI} * 2\right) * \left(\frac{1}{3} * 2\right) + \left(\frac{1}{2} * \frac{1}{EI} * 2\right) * \left(\frac{1}{3} * 2\right) - \left(\frac{1 \times 2}{2 \;\; 3} * 4 * \frac{2}{EI}\right) * \frac{3*2}{4}$

$= \frac{58}{3EI}$.

also;

ΔD = B.M. at D' in C.B.

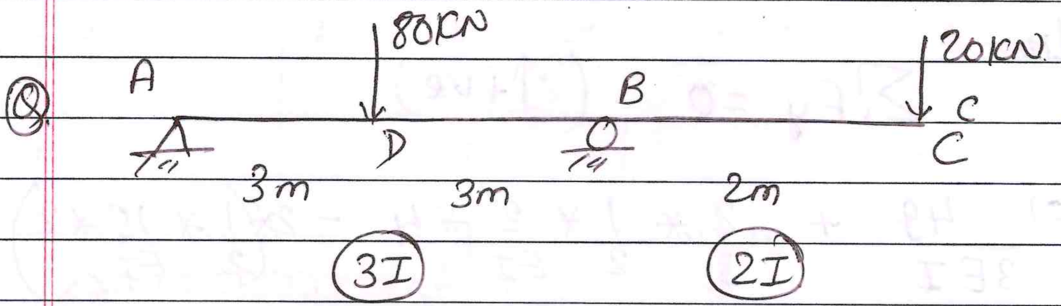
$$= \frac{49}{3EI} * 6 + \left(\frac{1}{2} * \frac{2}{EI} * 4\right) * \left(2 + \frac{1}{3} * 4\right) +$$

$$\left(\frac{1}{2} * \frac{4}{EI} * 2\right) * \left(\frac{2}{3} * 2\right) - \left(\frac{2}{3} * 4 * \frac{2}{EI}\right) * 4$$

$$- 2 * \left(\frac{1}{2} * \frac{15}{EI} * 2\right) * 4$$

$$= \frac{-74}{3EI}$$

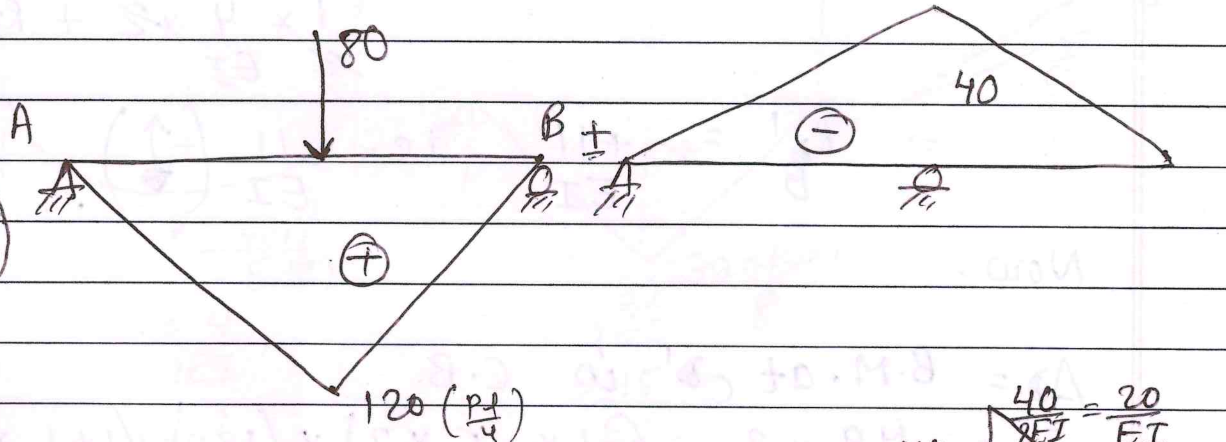
Q.



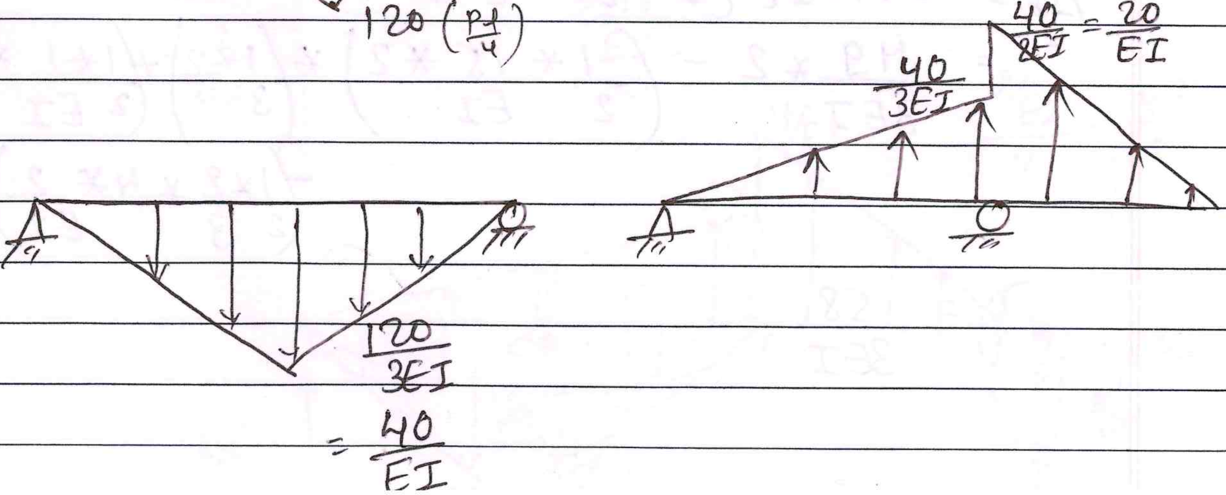
$\theta_A, \theta_B, \theta_C, \Delta_C = ?$

$\Delta_D, \theta_D = ?$

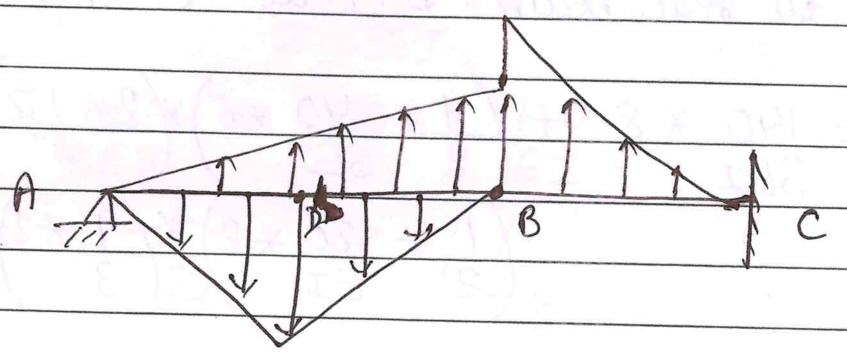
M diagram



M/EI diagram



∴ Conjugate beam (C.B.):



For reaction:

$$\sum M_B = 0 \quad (\circlearrowright +ve)$$

$$\Rightarrow -R_A * 6 + \cancel{2*}\left(\frac{1}{2} * \frac{40}{EI} * 3\right) * 3 - \left(\frac{1}{2} * \frac{40 * 6}{3EI}\right) * \frac{1}{3} * 6 = 0$$

$$\Rightarrow R_A = \cancel{\frac{-20}{EI}} \quad \frac{140}{3EI}$$

also;

$$R_A + R_C \cancel{=0} - \left(\frac{1}{2} * \frac{40}{EI} * 3\right) + \left(\frac{1}{2} * \frac{40}{3EI} * 6\right) + \left(\frac{1}{2} * \frac{20}{EI} * 2\right) = 0$$

$$\Rightarrow R_A + R_C = \frac{60}{EI}$$

$$\Rightarrow \frac{140}{3EI} + R_C = \frac{60}{EI}$$

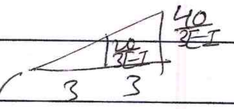
$$\Rightarrow R_C = \frac{40}{3EI}$$

Now:

ΔC in real beam = B.M. at C in C.B.

$$= \frac{140}{3EI} * 8 + \left(\frac{1}{2} * \frac{40}{3EI} * 6\right) * \left(2 + \frac{1 \cdot \cancel{6}^{2}}{\cancel{3}}\right) + \left(\frac{1}{2} * \frac{20}{EI} * 2\right) * \left(\frac{2}{3} * 2\right) - 2 * \left(\frac{1}{2} * \frac{40}{EI} * 3\right) * (2+3)$$

$$= \frac{-40}{EI} \quad \text{i.e. } (\downarrow)$$



Δ_D in real beam = B.M. at D in C.B

$$= \frac{140}{\cancel{3}EI} * \cancel{3} + \left(\frac{1}{2} * \frac{\cancel{20}^{10}}{\cancel{3}EI} * \cancel{3}\right) * \frac{1 * \cancel{3}}{\cancel{3}} - \left(\frac{1}{2} * \frac{\cancel{20}^{10}}{EI} * 3\right) * \left(\frac{1 * \cancel{3}}{\cancel{3}}\right)$$

$$= \frac{140}{EI} + \frac{10}{EI} - \frac{30}{EI}$$

$$= \frac{120}{EI} \quad \text{i.e. } (\downarrow)$$

θ_A = S.F. at A in C.B.

$$= \frac{140}{3EI}\,c$$

θ_D = S.F. at D in C.B.

$$= \frac{140}{3EI} + \left(\frac{1}{2} * \frac{\cancel{20}^{10}}{\cancel{3}EI} * \cancel{3}\right) - \left(\frac{1}{2} * \frac{\cancel{40}^{20}}{EI} * 3\right)$$

$$= \frac{140}{3EI} + \frac{10}{EI} - \frac{60}{EI}$$

$$= \frac{-10}{3EI} \quad \text{i.e.}$$

θ_B = S.F. at B in C.B.

$$= \frac{140}{3EI} + \frac{1}{2} * \frac{40}{3EI} * 6 - 2 * \left(\frac{1}{2} * \frac{40}{EI} * 3\right)$$

$$= \frac{140}{3EI} + \frac{40}{EI} - \frac{120}{EI}$$

$$= \frac{-100}{3EI} \quad \text{i.e.}$$

θ_C = S.F. at C in C.B.

$$= \frac{140}{3EI} + \frac{1}{2} * \frac{40}{3EI} * 6 - 2 * \left(\frac{1}{2} * \frac{40}{EI} * 3\right) + \frac{1}{2} * 2 * \frac{20}{EI}$$

$$= \frac{-100}{3EI} + \frac{20}{EI}$$

$$= \frac{-40}{3EI} \quad \text{i.e.}$$

CHAPTER – 5

INFLUENCE LINE DIAGRAM

In case of static loads; we calculate the reactions and stresses resultants (S.F., B.M, etc) by using the principle of statics. But in case of moving loads (Eg. Vehicles in bridge); the value of such stress resultants (S.F., B.M., etc.) vary as the position of load changes. Hence; we need to use the concept of influence lines for calculating reactions, S.F., B.M., etc. if moving loads are applied.

Influence line diagrams is a curve,

the ordinate of which at any point indicates the value of particular stress resultant (reaction S.F., B.M, etc.) at that point/section when unit (1 KN) load is moving on the span.

Uses of ILD:

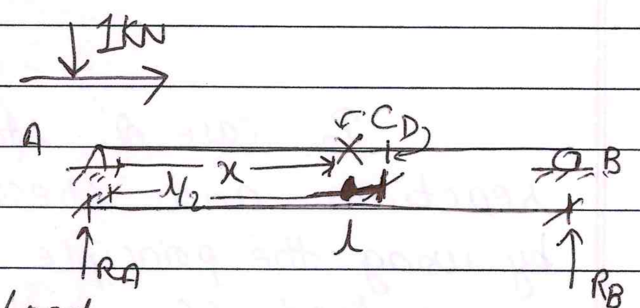
① It is used to calculate stress resultants for any position of load.

② It is used to know the position of loads which causes maximum shear force and bending moment at any section.

③ It is used to know the position of loads which causes absolute maximum S.F. and B.M. anywhere in the grider.

Influence line diagram can be plotted for various stress resultants like reactions, S.F., B.M., torsion, etc. at any section.

① I.L.D. for simply supported:



Let us consider a simply supported beam in which 1KN load is moving from left to right as shown in figure. Let 'x' be the distance of moving load from left support.

ⓐ I.L.D. for Ra:

$\sum M_A = 0$ (↻+ve)

$\Rightarrow 1 * x - R_B * l = 0$

$\Rightarrow R_B = \dfrac{x}{l}$ (linear variation)

$\sum V = 0$ (↑+ve)

$\Rightarrow R_A - 1 + \dfrac{x}{l} = 0$

$\Rightarrow R_A = \dfrac{l-x}{l}$ (linear variation)

If $x=0$ (at A); (i.e. when 1 KN is at 'A');
then $R_A = 1$ KN

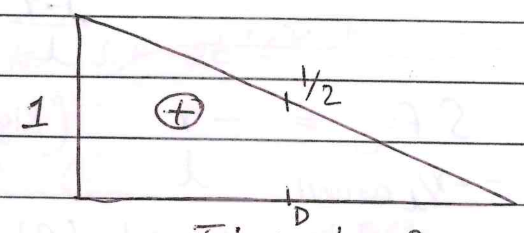
When 1 KN is at 'D'; $x = \dfrac{l}{2}$;

$R_A = \dfrac{l - l/2}{l} = \dfrac{1}{2}$ KN

When 1 KN is at B i.e. $x = l$;

$R_A = 0$

so,



I.L.D. for Ra

ⓑ I.L.D. for R_B:

When 1 KN is at 'A' (i.e. $x=0$)

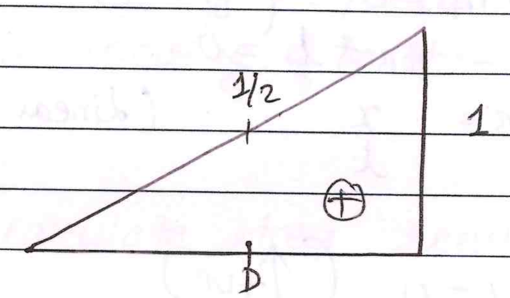
$\therefore R_B = 0$

When 1 KN is at 'D' (i.e. $x = l/2$);

$R_B = \dfrac{1}{2}$

when 1 KN is at 'B'; $x = l$

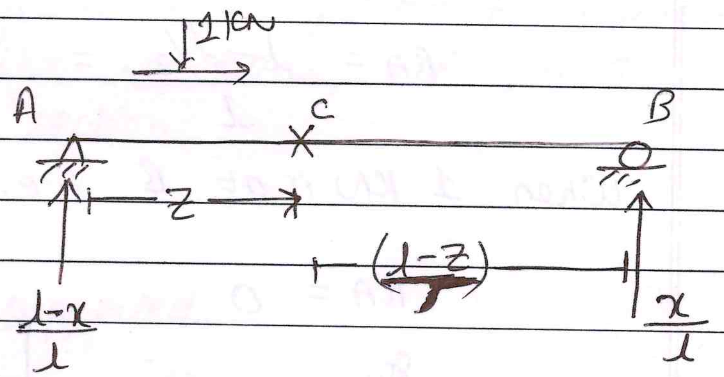
$\therefore R_B = 1.$



I.L.D for R_B

(c) I.L.D. for shear force at 'C':

Let us consider a section 'C' at 'Z' distance from 'A'. When 1 KN is on AC portion: $(0 \le x \le Z)$



$S.F._C = \frac{-x}{l}$ (Right ↑ is -ve)

Can take $S.F_C = \frac{l-x}{l} - 1 = -x/l$ as well

When 1 KN is at 'A';

$x = 0.$

$\therefore S.F._C = 0$

When 1 KN is at just left of 'C'; $x = Z$

$S.F._C = \frac{-Z}{l}$

When 1 KN is on CB portion

$S.F._C = \frac{l-x}{l}$ ← can take $-\left(\frac{x}{l} - 1\right)$ also same result

(def ↑ is +ve)

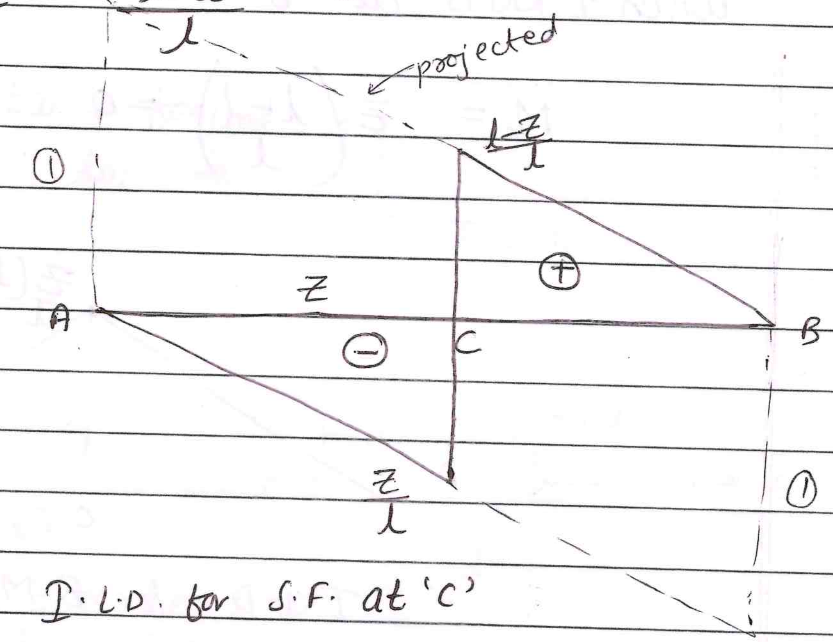
When 1 KN is just right of 'C';

$x = Z.$

$\therefore S.F._C = \frac{l-Z}{l}$

When 1 KN is at 'B';

$S.F._C = \frac{l-l}{l} = 0.$

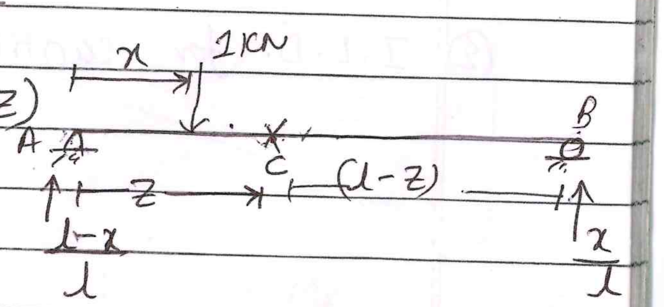


I.L.D. for S.F. at 'C'

(d) I.L.D. for B.M. at 'C':

When 1 KN is on AC $(0 < x < Z)$

$M_C = \frac{x}{l}(l-Z)$

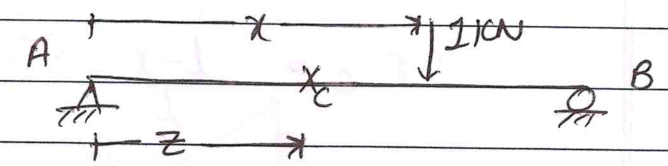


When 1 KN is at 'A' i.e. $x = 0$

$M_C = 0.$

When 1 KN is at 'C' i.e. $x = Z$

when 1 KN is on CB $(Z < x < l)$



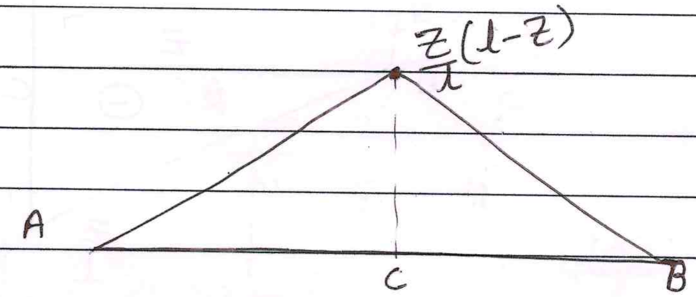
$$M_C = \left(\frac{l-x}{l}\right) Z$$

when 1 KN is at 'C' i.e. $x = Z$;

$$M_C = Z\left(\frac{l-Z}{l}\right)$$

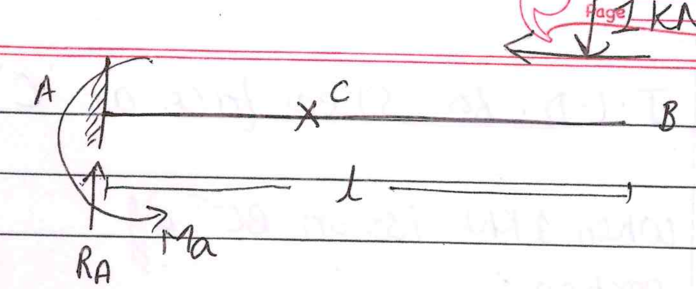
when 1 KN is at 'B' i.e. $x = l$;

$$M_C = Z\left(\frac{l-l}{l}\right) = 0$$



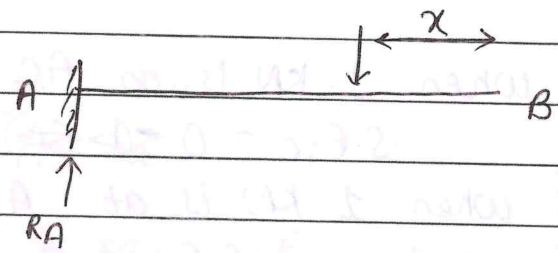
I.L.D. for B.M. at 'C'.

② I.L.D. for cantilever beam:



Let us consider a cantilever beam of length 'l' in which 1 KN is moving from right to left.

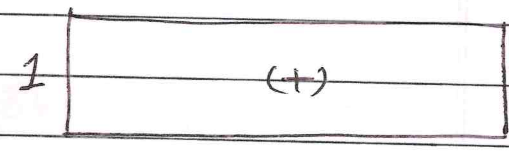
I.L.D. for R_A:



$$\Sigma V = 0$$

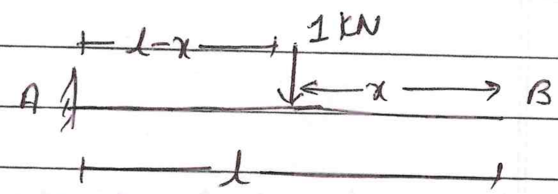
$$R_A = 1$$

$R_A = 1$ KN (constant) for load on every section. So; ⇒



I.L.D. for R_A

I.L.D. for M_A:



From fig;

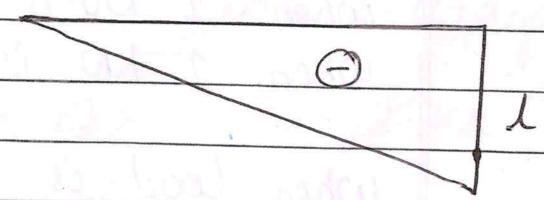
$$M_A = -(l-x)$$

when 1 KN is at 'B' i.e. $x=0$

$$M_A = -l$$

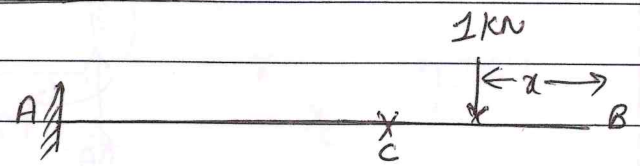
When 1 KN is at 'A' i.e. $x = l$

$$M_A = 0$$



I.L.D. for M_A

I.L.D. for Shear force at 'C' :



When 1 KN is on BC portion:

$S.F._C = -1$ KN

When 1 KN is on B ; $S.F._C = 1$ KN

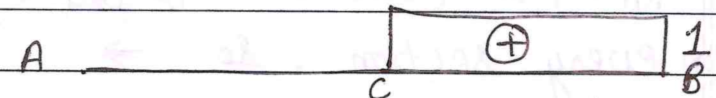
When 1 KN is just right at c = 1 KN

When 1 KN is on AC portion.

$S.F._C = 0$ ~~1 KN~~

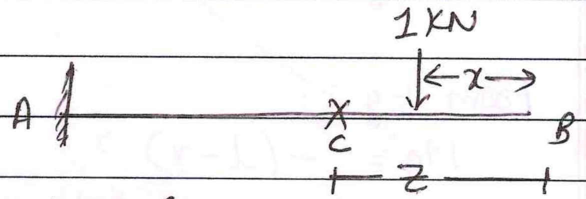
When 1 KN is at A;

$S.F._C = 0$



I.L.D. for S.F. at 'C'.

I.L.D. for B.M. at 'C' :



When Load is at BC portion: $(0 < x < z)$

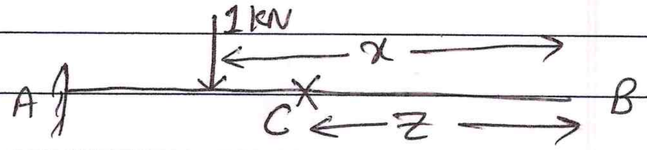
$M_C = -(z-x)$

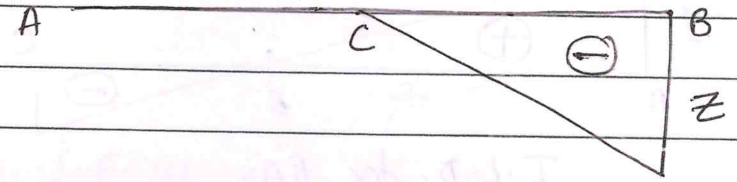
When 1 KN is at 'B'; $x = 0$. $\therefore M_C = -z$

When 1 KN is at 'c'; $x = z$ $M_C = -(z-z) = 0$.

When Load is at CA portion

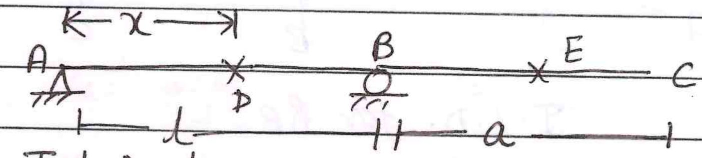
$M_C = 0$ (at every place)





I.L.D. for ~~S.F.~~ B.M. at 'C'.

H/W



Draw I.L.D. for

R_A, R_B, $S.F._D$, M_D, $S.F._E$, M_E

@ I.L.D. for Reaction:

~~When load is at AB;~~

$\sum_i M_A = 0$ (↺ +ve)

$\Rightarrow R_B * l - 1 * x = 0$

$\Rightarrow R_B = \dfrac{x}{l}$ ~~when~~

$R_A + R_B = 1$

$\Rightarrow R_A + \dfrac{x}{l} = 1$

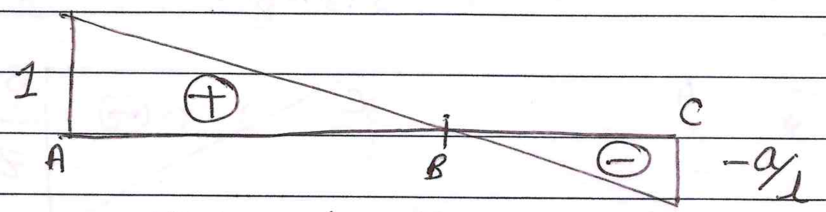
$\Rightarrow R_A = \dfrac{l-x}{l}$

For R_A: when $x = 0$ (at A); $R_A = 1$ when $x = (l+a)$ (at free end);

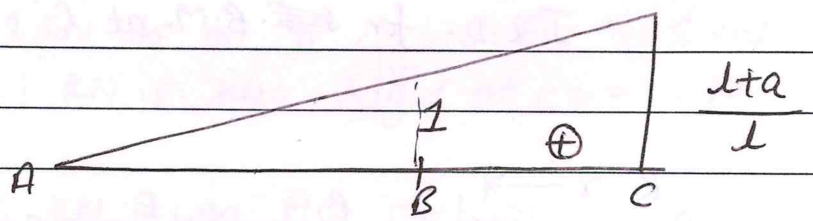
when $x = l$ (at B); $R_A = 0$ $R_A = \dfrac{\cancel{l} - \cancel{l} - a}{l} = \dfrac{-a}{l}$

For R_B: when $x = 0$ (at A); $R_B = 0$ when $x = (l+a)$ (at free end)

when $x = l$ (at B); $R_B = 1$ $R_B = \dfrac{l+a}{l}$



I.L.D. for R_A

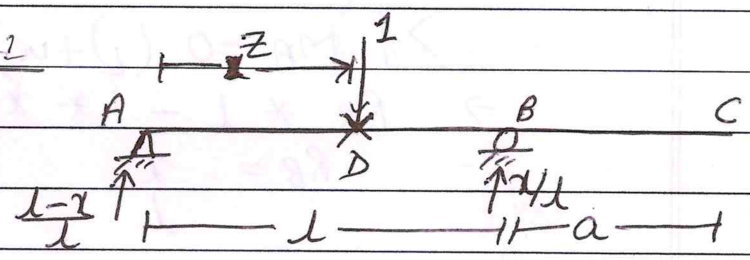


I.L.D. for R_B

(b) I.L.D. for shear force at 'D' :

~~S.F.~~

~~(a) at AB section:~~



~~when load is at AD section: $(0 \le x \le l)$~~

$S.F._{CD} = \frac{l-x}{l} - 1 = \frac{-x}{l}$

When 1 KN is at A ; $x = 0$.

$\therefore S.F._{CD} = 0$

When 1 KN is at B ; $x = l$.

$\therefore S.F._D = -1$

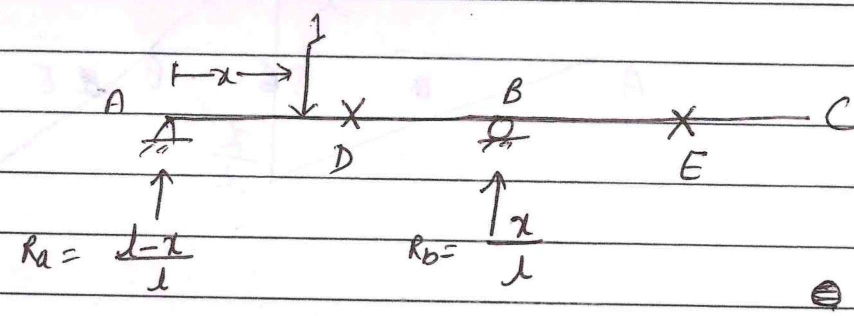
When 1 KN is at mid ; $x = l/2$

$\therefore S.F._D = \frac{-1}{2}$

At

~~When load is at BC section:~~

I.L.D. for S.F. at 'E'



When load at AB section:

$S.F._E = 0$

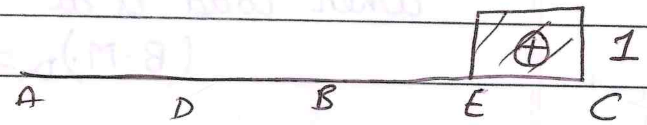
When load at BC section

when just right of E;

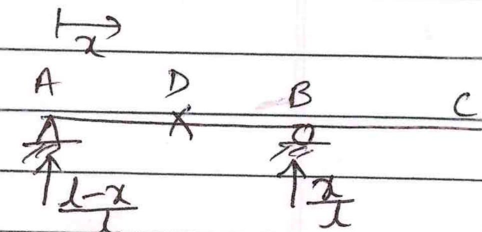
$S.F._E = 1$

When just left of E;

$S.F._E = 0$.



I.L.D. for S.F. at 'D':



When load is at AB section:

When just left of D:

$S.F._D = \frac{-x}{l}$ (~~Right~~ or $\frac{l-x-l}{l}$)

at A ; $x = 0$

$\therefore S.F._D = 0$

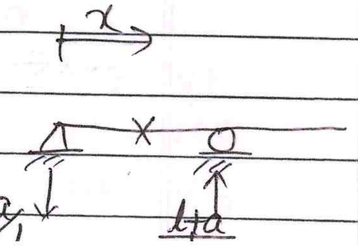
~~At~~ at B : $x = l$

$\therefore S.F._D = -\frac{l-l}{l} = 0$

When just right of D;

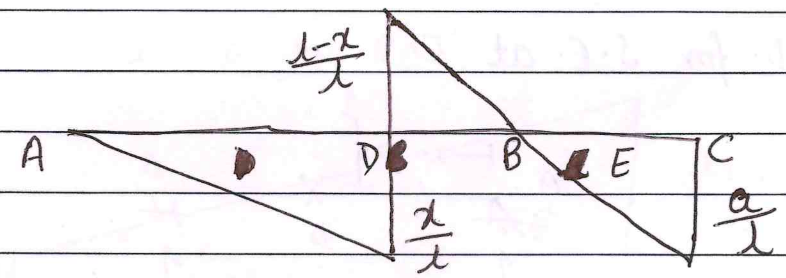
$S.F._D = \frac{l-x}{l}$

When load is at BC section:



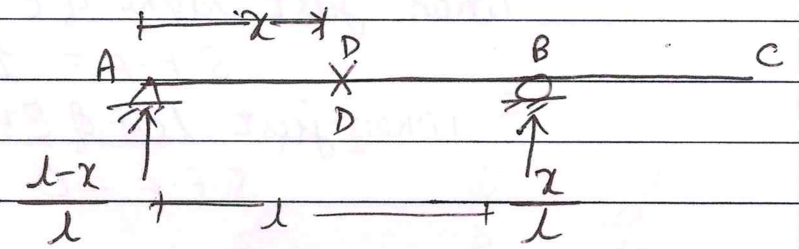
~~S.F.~~ When just left of $D = -\left(\frac{l+a}{l}\right)$

when just right of D: $\frac{-a}{l}$



I. L. D. for B. M. at D:

When load is at AB section:



When load is at AB section;

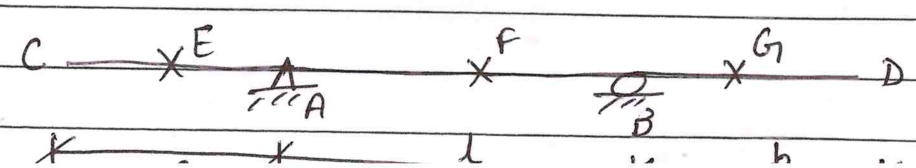
$$(B\cdot M)_D = \left(\frac{l-x}{l}\right)\cdot x$$

$\frac{l-x}{l}$ ·

x

~~When load is at B:~~

H.W. I.L.D. for double overhanging beam:



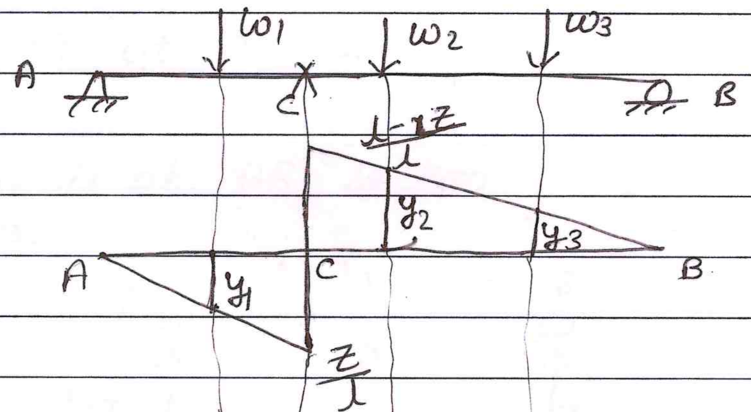
VE VF
ME MF
RA VG
RB MG

Use of I.L.D. :

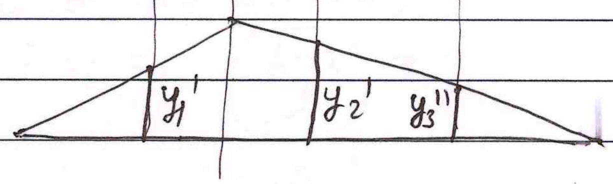
1. It is used to calculate the value of stress resultants at any section for given loading.

a) For point load :

In case of point loads; shear force and bending moments at any section is calculated by multiplying the intensity of load to the corresponding ordinate in I.L.D.

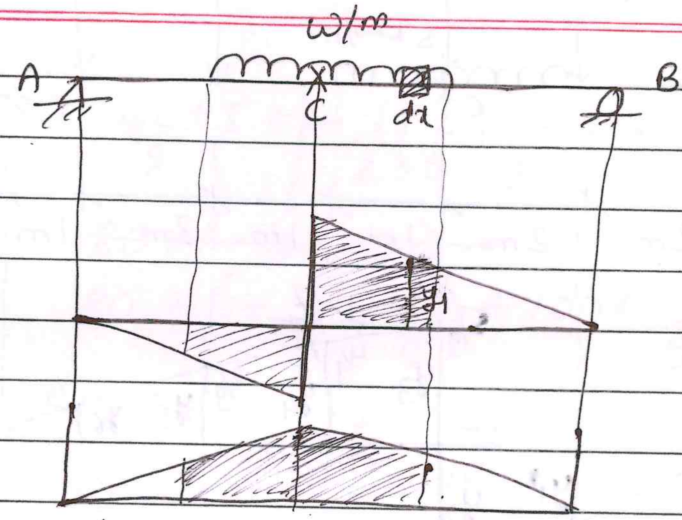


$S.F._C = -w_1 * y_1 + w_2 * y_2 + w_3 * y_3$



$B.M._C = w_1 * y_1' + w_2 * y_2' + w_3 * y_3'$

b) For UDL :



S.F. C under small element $dx = w * dx * y_1$

S.F. at C due to UDL from x_1 to x_2 $= \int_{x_1}^{x_2} w * dx * y_1$

→ S.F. at C due to UDL from x_1 to x_2 = w * area of I.L.D. covered by UDL

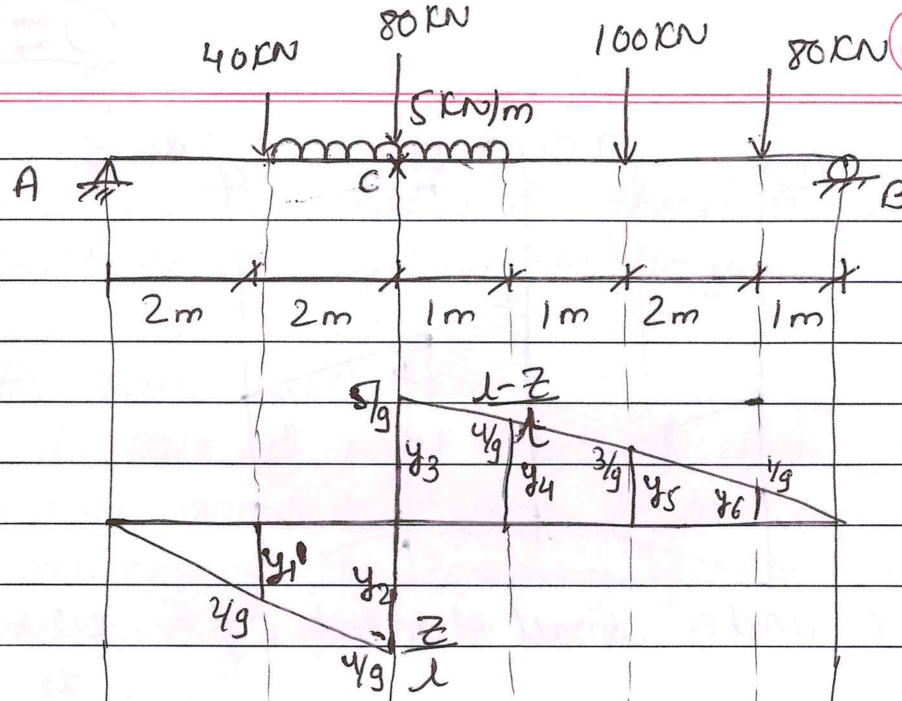
Similarly;

$B.M._C$ = w * area of I.L.D. for B.M. at C covered by UDL

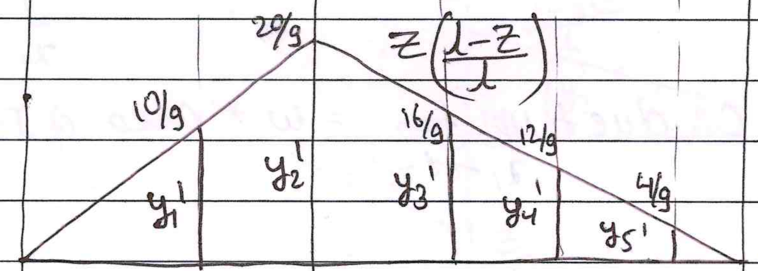
Q. use I.L.D. to calculate shear force and bending moment at 4m from A.

P.T.O. ⇒

11



I.L.D. for S.F. at 'C'



I.L.D. for B.M. at 'C'

$Z = 4m$

$l = 9m$

$\frac{Z}{l} = \frac{4}{9}$

$\frac{l-Z}{l} = \frac{5}{9}$

$Z\left(\frac{l-Z}{l}\right) = 4 * \frac{5}{9} = \frac{20}{9}$

So;

$$S.F._C = -40 * \frac{2}{9} + 5 * \left[-\frac{1}{2}\left(\frac{2}{9}+\frac{4}{9}\right)\cdot 2 + \frac{1}{2}\left(\frac{5}{9}+\frac{4}{9}\right)\cdot 1\right] +$$

$$\cancel{80 * (-y_2 + y_3) +}$$

$$80 * \left(-\frac{4}{9} + \frac{5}{9}\right) + 100 * \frac{3}{9} + 80 * \frac{1}{9}$$

$$= 41.38 \quad \text{i.e. } \frac{745}{18}$$

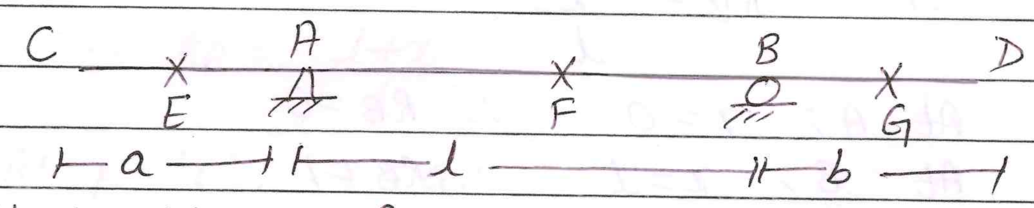
$$B.M._C \Rightarrow \cancel{40 * y_1'}$$

$$= 40 * \frac{10}{9} + 5 * \left[\frac{1}{2} * \left(\frac{10}{9} + \frac{20}{9}\right)\cdot 2 + \frac{1}{2} * \left(\frac{20}{9} + \frac{16}{9}\right)\cdot 1\right]$$

$$+ 80 * \frac{20}{9} + 100 * \frac{12}{9} + 80 * \frac{4}{9}$$

$$= 417.78 \quad \text{i.e. } \frac{3760}{9}$$

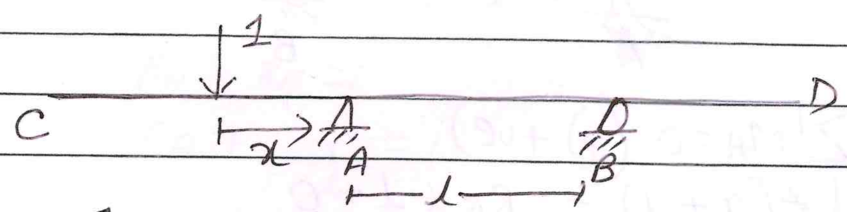
H/W Q/N



$V_E, V_F, M_F, R_A, R_B, V_G = ?$

I.L.D. for R_B:

When load is at AC portion:



$\sum M_A = 0$ (↻ +ve)

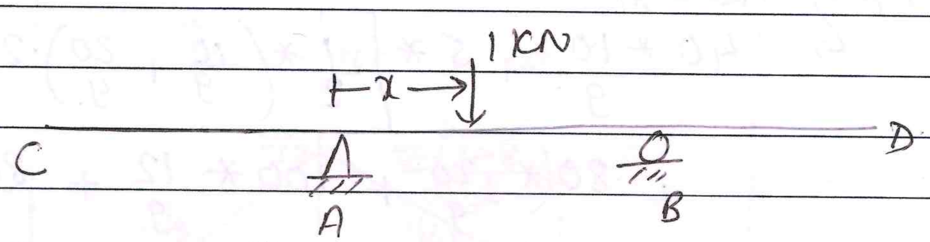
$\Rightarrow 1 * x + R_B * l = 0$

$\Rightarrow R_B = \dfrac{-x}{l}$

So; At C: $x = a$ $\therefore R_B = \dfrac{-a}{l}$

at A: $x = 0$ $\therefore R_B = 0$

When load is at AB portion:



$\sum M_A = 0$ (↻ +ve)

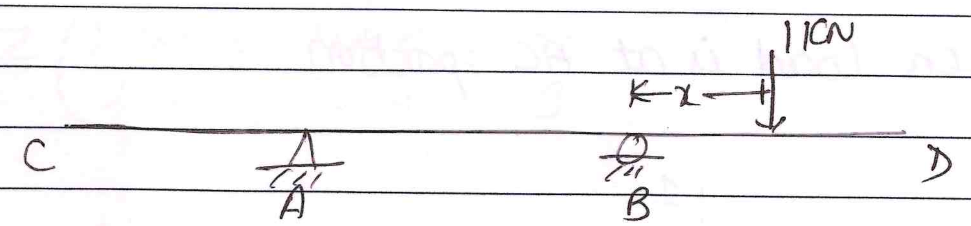
$\Rightarrow 1 * x - R_B * l = 0$

$\Rightarrow R_B = \dfrac{x}{l}$

At A; $x = 0$ $\therefore R_B = 0$

At B; $x = l$ $\therefore R_B = 1$

When load is at BD portion



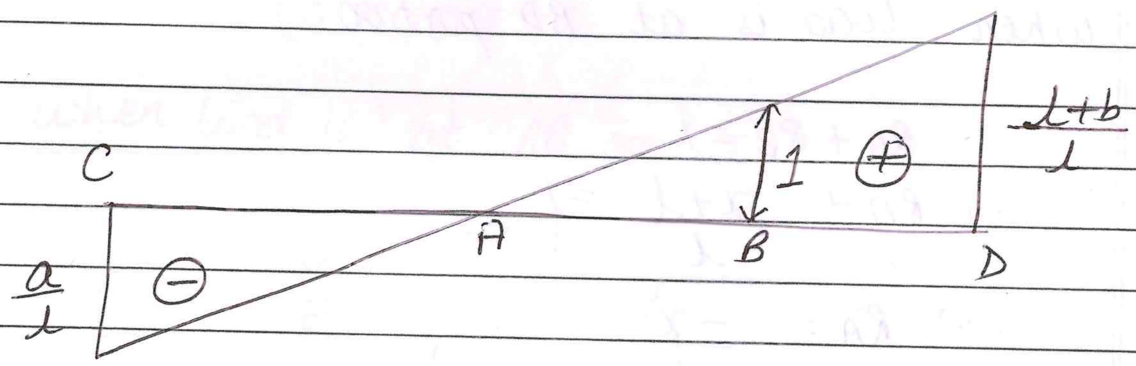
$\sum M_A = 0$ (↻ +ve)

$\Rightarrow 1 * (x + l) - R_B * l = 0$

$\Rightarrow R_B = \dfrac{x + l}{l}$

At B; $x = 0$ $\therefore R_B = 1$

at D; $x = b$ $\therefore R_B = \dfrac{l + b}{l}$



For I.L.D. for R_A:

When load is at AC portion:

$R_A + R_B = 1$

$\Rightarrow R_A - \dfrac{x}{l} = 1$

$\Rightarrow R_A = \dfrac{l + x}{l}$

At pt. 'C'; $x = a$

$\therefore R_A = \dfrac{l + a}{l}$

At pt. 'A'; $x = 0$

$\therefore R_A = 1$

When load is at AB portion:

$R_A + R_B = 1$

$\Rightarrow R_A + \dfrac{x}{l} = 1$

$\Rightarrow R_A = \dfrac{l - x}{l}$

At pt. 'A' ; $x=0$ $\therefore R_A = 1$
At pt. 'B' ; $x=l$ $\therefore R_B = 0$

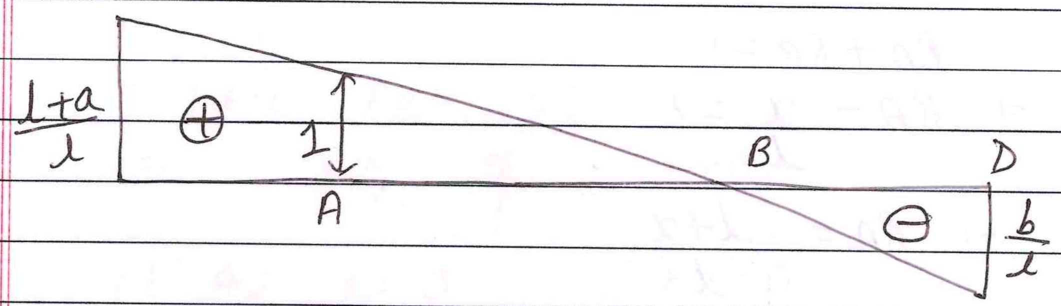
when load is at BD portion:

$$R_A + R_B = 1$$

$$\Rightarrow R_A + \frac{x+l}{l} = 1$$

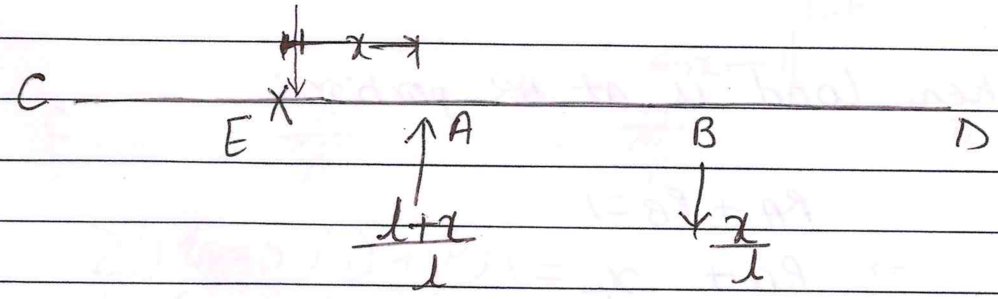
$$\Rightarrow R_A = \frac{-x}{l}$$

At B; $x=0$ $\therefore R_A = 0$
At D; $x=b$ $\therefore R_A = \frac{-b}{l}$



for I.L.D. for S.F.E :

when load is at AC portion:



when at just left of E;
$$(S.F.)_E = -1$$

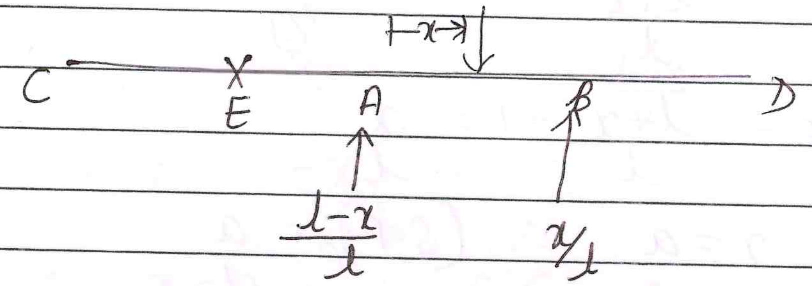
S.F. ↑ ↓ ↓ ↑

left ↑ Right ↓ Then +ve Then -ve

When at just right of E;
$$(S.F.)_E = 0$$

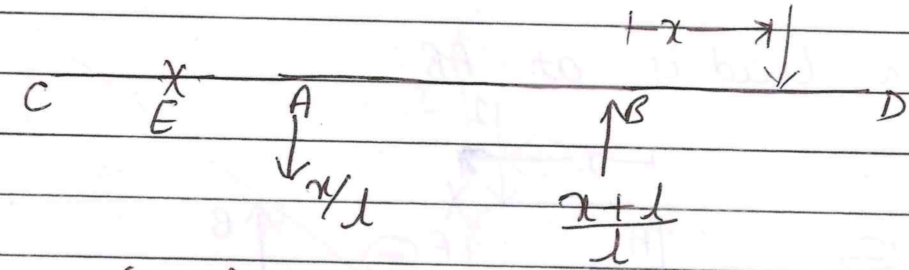
when load is at 'C';
$$(S.F.)_E = -1$$

when load is at AB portion:

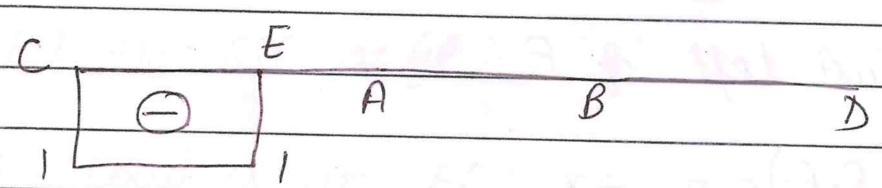


$$(S.F.)_E = 0$$

when load is at BD portion:

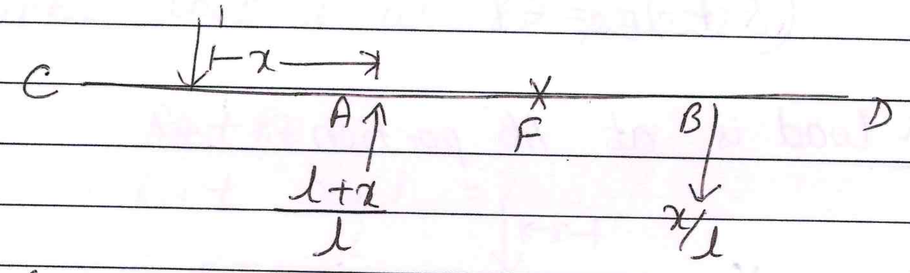


$$(S.F.)_E = 0$$



I.L.D. for $(S.F)_F$:

when load is at AC :

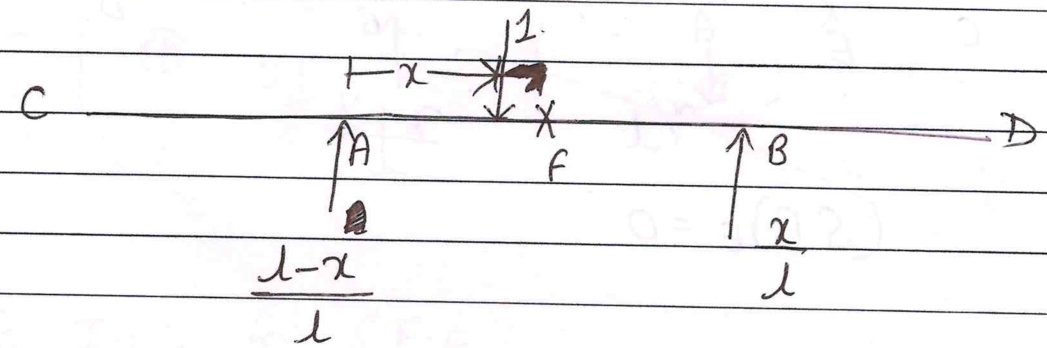


$(S.F)_F = \frac{l+x}{l} - 1 = \frac{x}{l}$

at C ; $x = a$ $\therefore (S.F)_F = \frac{a}{l}$

at A ; $x = 0$ $\therefore (S.F)_F = 0$

when load is at AB :



At just left of F :

$(S.F)_F = \frac{-x}{l}$

at A ; $x = 0$ $\therefore S.F. = 0$

at B ; $x = l$ $\therefore S.F. = -1$

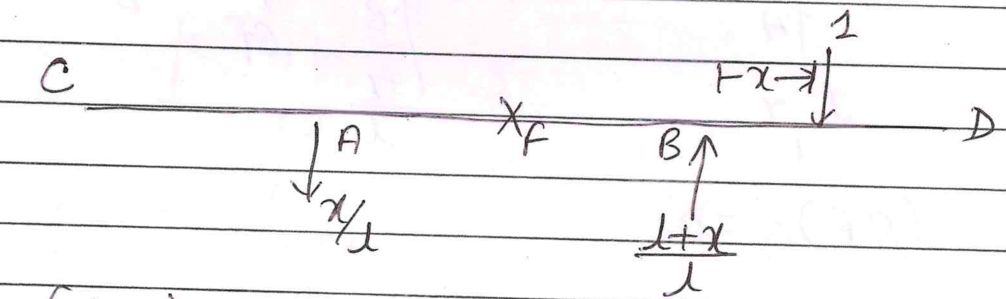
At just right of F ;

$(S.F)_F = \frac{l-x}{l}$

At A ; $x = 0$ $\therefore S.F. = 1$

at B ; $x = l$ $\therefore S.F. = 0$

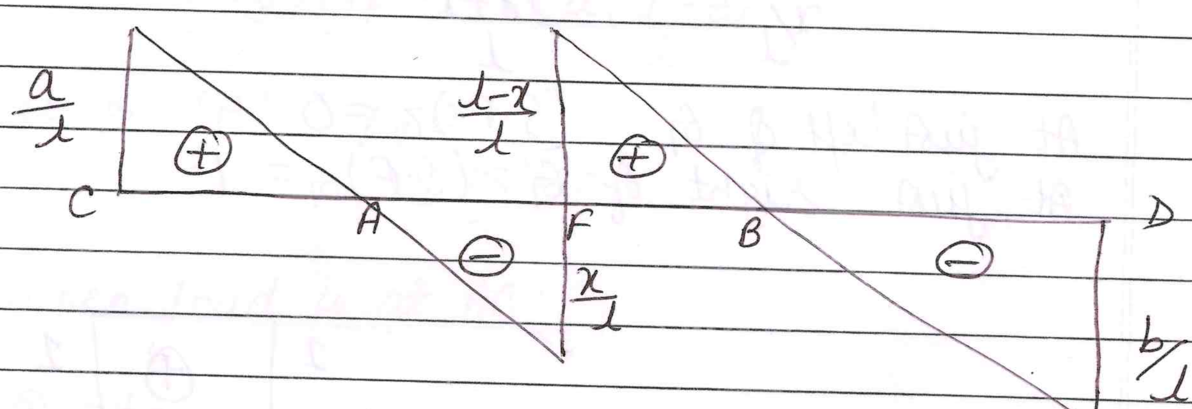
when load is at BD :



$(S.F)_F = \frac{-x}{l}$

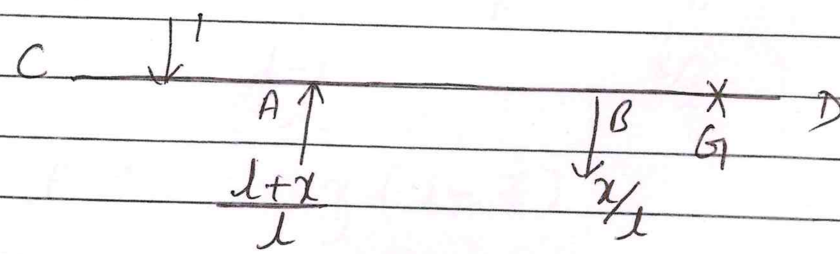
At B ; $x = 0$ $\therefore S.F. = 0$

At D ; $x = b$ $\therefore S.F. = \frac{-b}{l}$



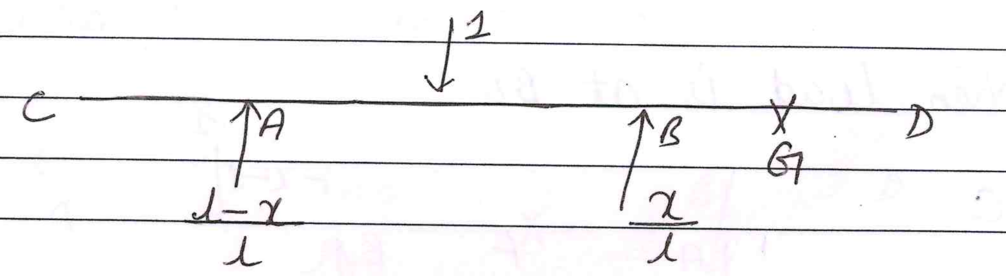
I.L.D. for S.F. at G :

when load is at AC portion :



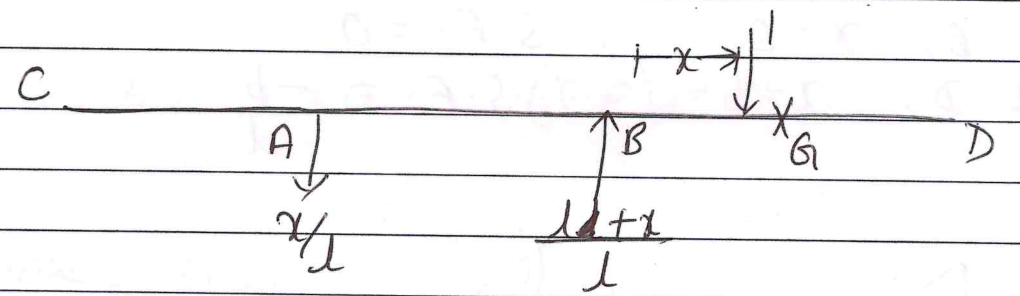
$(S.F)_G = 0$

When load is at AB portion

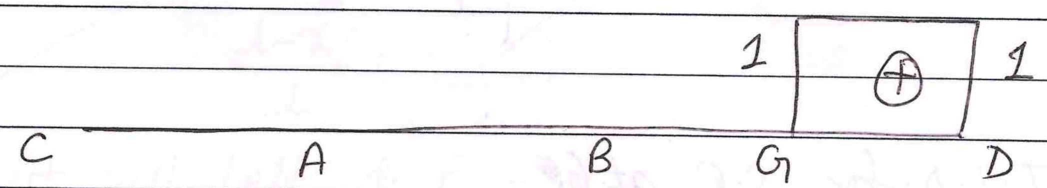


$(S.F.)_G = 0$

When load is at BD portion:

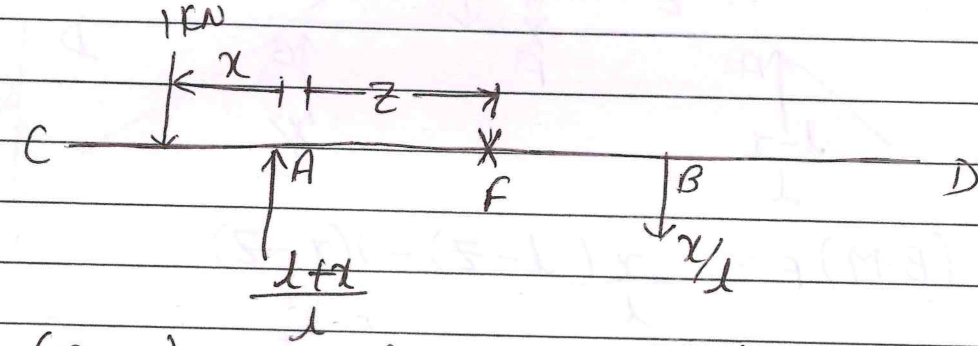


At just left of G; $(S.F.)_G = 0$
At just right of G; $(S.F.)_G = 1$



I.L.D. for B.M. at F:

When load is at AC portion:



$$(B.M.)_F = -(x+z) + \left(\frac{l+x}{l}\right) \cdot z$$

$$= \frac{-x(l-z)}{l}$$

At pt. 'C'; $x = a$

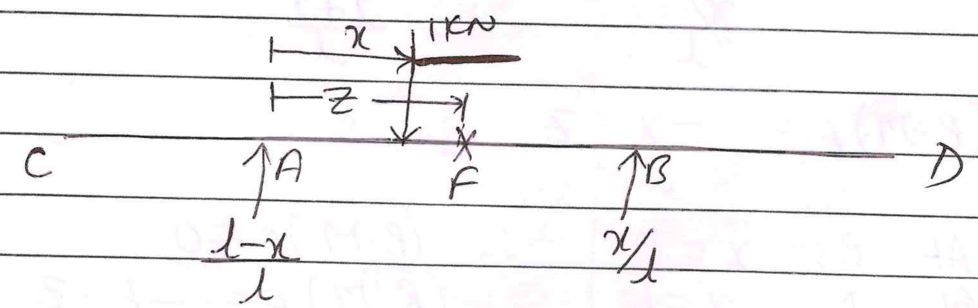
$$\therefore (B.M.)_F = \frac{-a(l-z)}{l}$$

at pt. 'A'; $x = 0$

$$\therefore (B.M.)_F = 0$$

When load is at AB:

(a) When load is at AF portion:



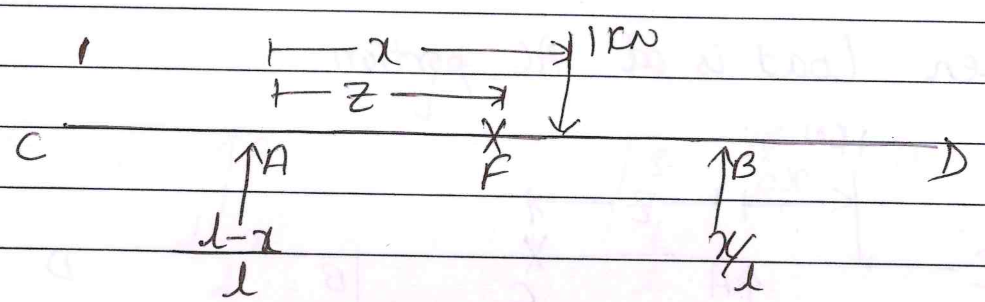
$$(B.M.)_F = \frac{x}{l}(l-z)$$

At A; $x = 0$ $\therefore (B.M.)_F = 0$

At F; $x = z$ $\therefore (B.M.)_F = \frac{z}{l}(l-z)$

जुन pt · बाट x लिएको छ Rxn निकाल्ने बेला, पछिपनि load त्यही segment मा पर्दा x त्यसबाटै लिने !!!

Ⓑ when load is at BF portion:



$$(B.M.)_F = \frac{x}{l}(l-z) - 1(x-z)$$

$$= \frac{z(l-x)}{l}$$

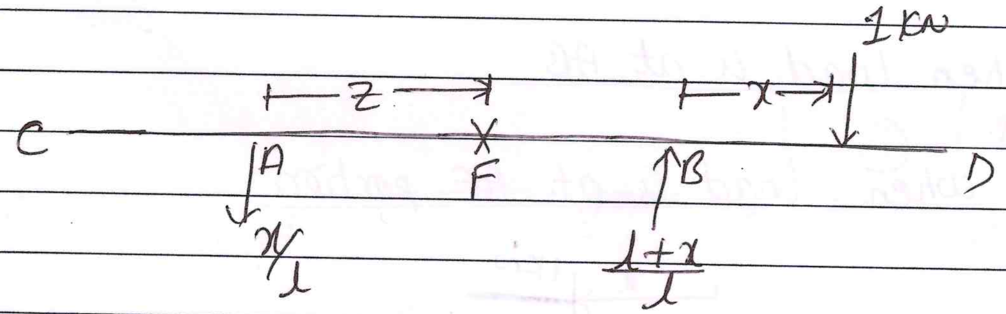
at B; $x = l$

$\therefore (B.M.)_F = 0$

at F; $x = z$

$$\therefore (B.M.)_F = \frac{z(l-z)}{l}$$

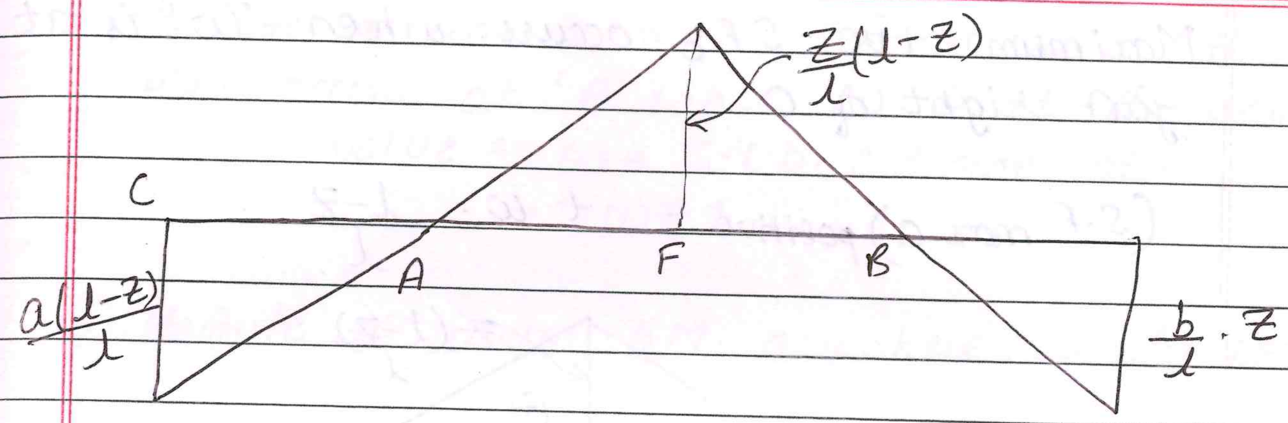
when load is at BD:



$$(B.M)_F = \frac{-x}{l}\cdot z$$

At B; $x = 0$ $\therefore (B.M.)_F = 0$

At D; $x = b$ $\therefore (B.M)_F = \frac{-b}{l}\cdot z$



Uses of I.L.D (ctd...):

② To find the position of load for maximum S.F. and B.M. for various load cases:

Ⓐ single point load.

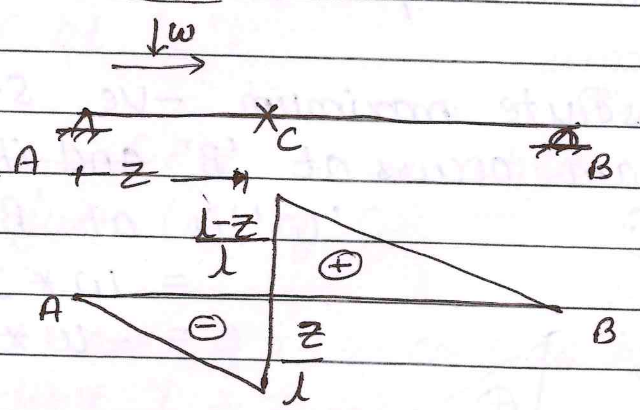
Ⓑ. UDL larger than the span.

Ⓒ. UDL shorter than the span.

Ⓓ. Train of concentrated load.

Ⓐ. Single point load:

ⓐ Maximum S.F. at C

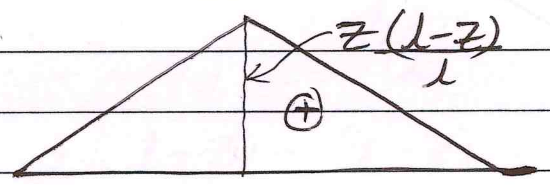


From I.L.D. for S.F. at 'C', maximum negative S.F. at 'C' occurs when 'w' is just left of C

$$(S.F._{max,c}) = -w.\frac{z}{}$$

Maximum +ve $S.F_C$ occurs when 'w' is at just right of C.

$$(S.F._{max\ c})_{positive} = +w \cdot \frac{l-z}{l}$$



ⓑ Maximum B.M. at C:

From I.L.D. for B.M. at 'C'; maximum B.M. at 'C' occurs when 'w' is at C.

$$(B.M._{max,c}) = w * \text{ILD Ordinate at C}$$
$$= w \cdot \frac{z(l-z)}{l}$$

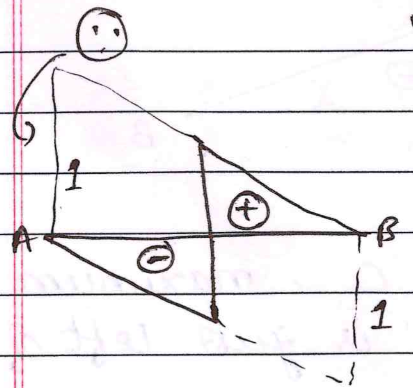
ⓒ. Absolute maximum S.F. anywhere in the beam:

It occurs where I.L.D. ordinate all over the span is maximum.

- Absolute maximum -ve S.F. occurs in the beam occurs at 'B' ~~and it's value is~~ when 'w' is at 'B' and it's value

$= w * \text{I.L.D. Ordinate at B}$

$= -w * 1$



- Absolute maximum +ve S.F. occurs in the beam occurs at 'A' when 'w' is at 'A' and it's value $= w * \text{I.L.D. Ordinate at A}$

$= w * 1$.

ⓓ. Absolute maximum B.M. anywhere in the beam:

It occurs at mid-span and when 'w' is at mid-span and it's value $= w * \text{I.L.D. ordinate at mid-span}$

$$= w * \frac{z(l-z)}{l}$$

At mid-span; $z = \frac{l}{2}$

$$= w * \frac{l/2\,(l - l/2)}{l}$$

$$= \frac{wl}{4}$$

Ⓑ. UDL is longer than the span:

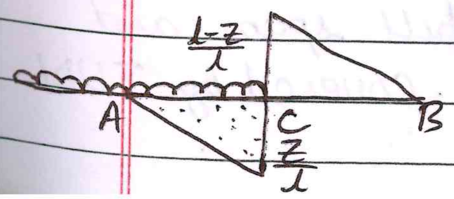
ⓐ. Maximum S.F. at 'C'

$d > l$

Maximum -ve $S.F._C$ occurs when head of the UDL is at just left of C.

$(S.F._{c,max}) = w * \text{area of I.L.D. covered by UDL}$ (i.e. betn A & C)

$$= -w * \frac{1}{2} * \frac{z}{l} * z$$

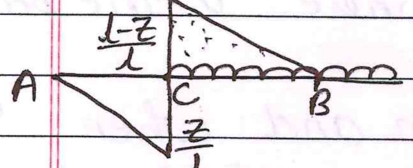


$$= -\frac{wz^2}{2l}$$

Maximum +ve S.F. occurs when tail of U.D.L. is just right of C

$$(S.F._{C,\,max})_{+ve} = w * \text{area of I.L.D covered by UDL}$$

$$= w * \frac{1}{2} * \left(\frac{l-z}{l}\right) * (l-z)$$



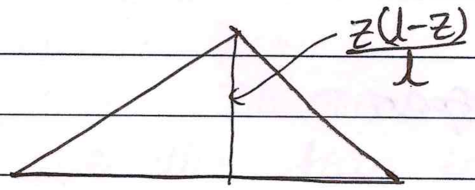
$$= \frac{w(l-z)^2}{2l}$$

(b). Maximum B.M. at C :

~~From~~ From I.L.D. for B.M. at 'C' ; maximum B.M. at 'C' occurs when UDL covers the whole span

$$(B.M._C)_{maximum} = w * \text{area of I.L.D. covered by UDL}$$

$$= w \cancel{2} * \frac{1}{2} \cdot \frac{z(l-z)}{\cancel{l}} * \frac{\cancel{l}}{\cancel{2}}$$



$$= \frac{w z (l-z)}{2}$$

(c). Absolute maximum S.F. anywhere in the beam:

It occurs when I.L.D. ordinate all over the span is maximum.

- Absolute maximum −ve S.F. occurs in the beam at 'B' when UDL covers full span and it's value = w * area of I.L.D. covered by UDL

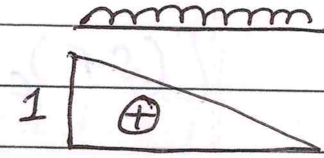
$$= -w * \left(\frac{1}{2} * 1 * l\right)$$

$$= -\frac{wl}{2}$$

- Absolute maximum +ve S.F. occurs in the beam at 'A' when 'w' covers entire span and it's value = w * area of I.L.D. covered by UDL

$$= w * \left(\frac{1}{2} * 1 * l\right)$$

$$= \frac{wl}{2}$$

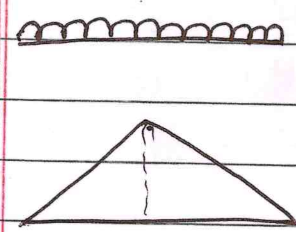


(d) Absolute maximum B.M. anywhere in the beam:

It occurs at mid-span when 'w' covers the entire span and it's value

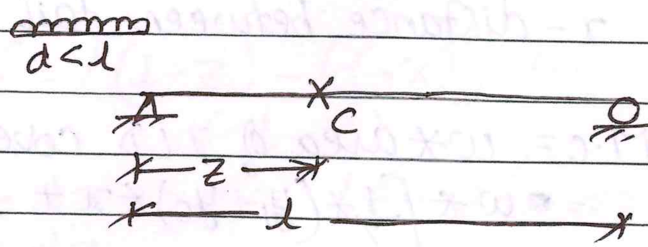
$$= w * \text{area of ILD covered by UDL}$$

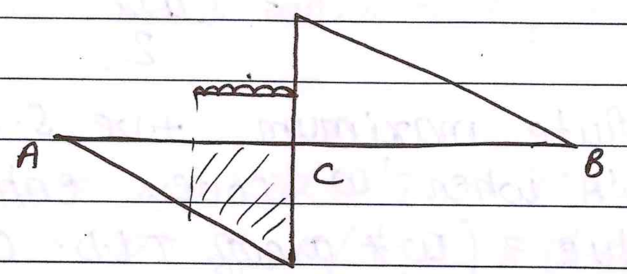
$$= w * \frac{1}{2} * \frac{z(l-z)}{l} * l \quad \left(\text{when } z = \frac{l}{2}\right)$$



$$= \frac{wl^2}{8} \quad \leftarrow \text{keeping } z = \frac{l}{2}$$

(e) UDL shorter than the span:





ⓐ Maximum S.F. at 'c':

- Maximum −ve S.F. at 'c' occurs when head of UDL reaches just left of C and it's value is:

$$[(S.F.)_{c\ max}]_{-ve} = w * \text{area of I.L.D. covered by UDL}$$

- Maximum +ve S.F. at 'c' occurs when ~~he~~ tail of UDL reaches just right of C and it's value is;

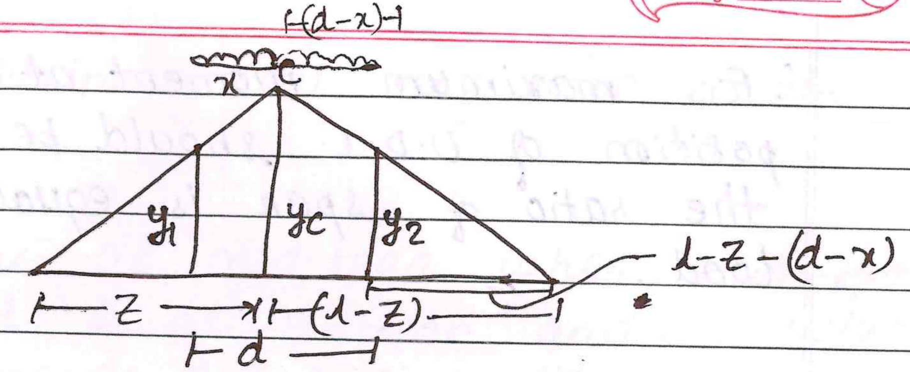
$$(S.F_{c}\ max)_{+ve} = w * \text{area of ILD covered by UDL}$$

ⓑ Maximum B.M. at 'c'

Maximum B.M. at 'c' occurs when some part of UDL is at left and some part of ~~t~~ UDL is at right.

Let x – distance between tail of UDL and 'c'.

$$B.M._c = w * \text{area of ILD covered by UDL}$$
$$= w * \left[\frac{1}{2} * (y_1 + y_c) * x + \frac{1}{2} * (y_c + y_2) * (d - x)\right]$$



for M_c to be maximum;

$$\frac{dM_c}{dx} = 0$$

$$\Rightarrow \frac{dM_c}{dx} = w * \left[\frac{y_1 + y_c}{2} - \left(\frac{y_c + y_2}{2}\right)\right]$$

$$\Rightarrow 0 = \frac{y_1}{2} + \cancel{\frac{y_c}{2}} - \cancel{\frac{y_c}{2}} - \frac{y_2}{2}$$

$$\Rightarrow \boxed{y_1 = y_2}$$

From I.L.D.;

$$\frac{y_1}{y_c} = \frac{z - x}{z}$$

$$\Rightarrow y_1 = \frac{z - x}{z} \cdot y_c$$

and;

$$\frac{y_2}{(l-z)-(d-x)} = \frac{y_c}{l - z}$$

$$\Rightarrow y_2 = \frac{[(l-z)-(d-x)]\,y_c}{(l-z)}$$

As $y_1 = y_2$;

$$\Rightarrow \frac{z - x}{z} \cdot \cancel{y_c} = \frac{[(l-z)-(d-x)]\,\cancel{y_c}}{(l-z)}$$

$$\Rightarrow \cancel{lz} - \cancel{z^2} - lx + \cancel{xz} = \cancel{lz} - \cancel{z^2} - dz + \cancel{xz}$$

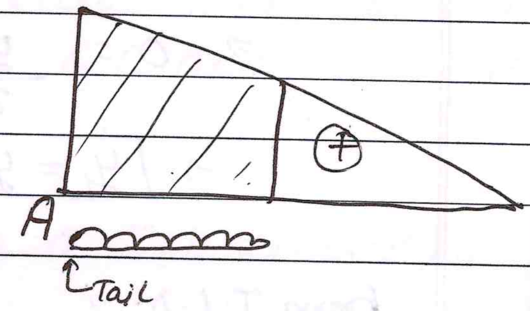
$$\Rightarrow lx = dz$$

$$\Rightarrow \boxed{\frac{z}{l} = \frac{x}{d}}$$

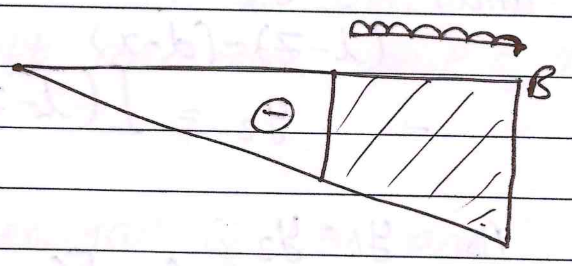
∴ For maximum moment at 'C'; the position of U.D.L. should be such that the ratio of span is equal to ratio of load.

Ⓒ Absolute maximum ~~+ve~~ S.F. ~~at~~ anywhere in beam:

- Absolute +ve max. S.F. occurs at 'A' when tail of UDL reaches 'A' and it's value is = w * area of ILD covered by UDL

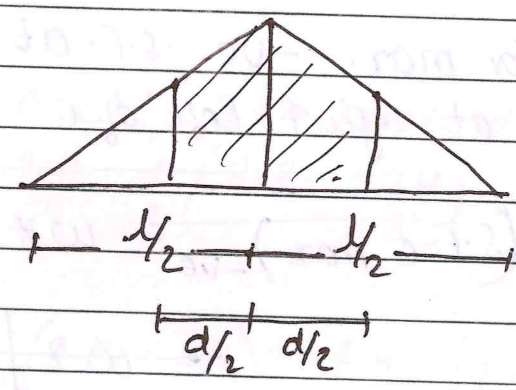


- Absolute maximum −ve S.F. occurs at 'B' when head of the UDL reaches 'B' and it's value is = w * area of ILD covered by UDL



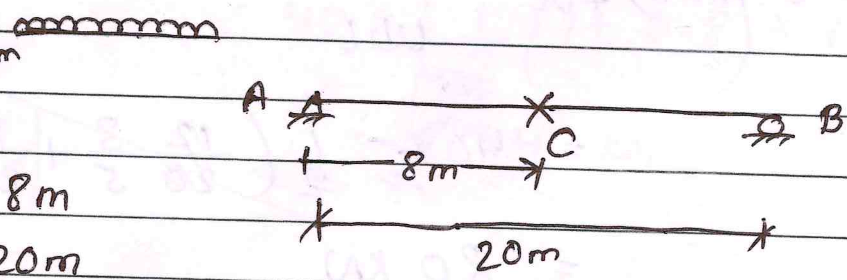
ⓓ. Absolute maximum B.M. anywhere in the beam:

It occurs at mid-span when mid-~~span~~ point of UDL is at mid-span and it's value is = w * area of ILD covered by UDL.



Q. A simply supported beam has span of 20 m subjected to UDL of 40 KN/m of 4m length. The load crosses left-to-right. Draw ILD for S.F. and B.M. at 8m from left support. Using I.L.D.; calculate max. S.F. & B.M. at 'C' and also calculate the position ~~of~~ and magnitude of absolute maximum S.F. and B.M.

Here;



$Z = 8m$

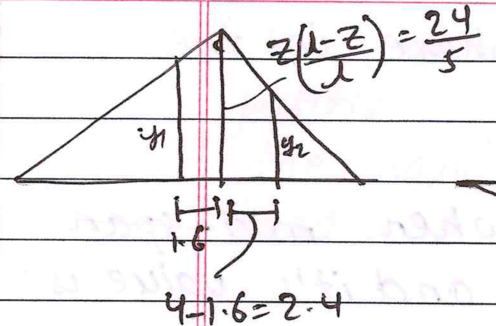
$l = 20m$

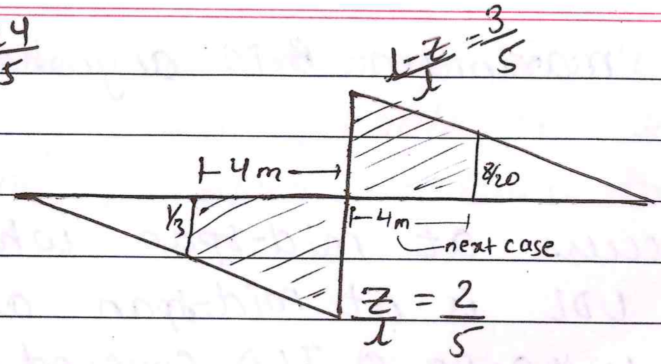
$d = 4m$

$W = 40 \text{ KN/m}$

$\frac{Z}{l} = \frac{8}{20} = \frac{2}{5}$ $\qquad Z\left(\frac{l-Z}{l}\right) = \frac{8*3}{5} = \frac{24}{5}$

$l-Z = 20-8 = 12 = 3$





For max. -ve S.F. at 'C'; head of UDL should be at just left of C.

$$\therefore (S.F._{C\,max})_{-ve} = w * \text{area of ILD covered by UDL}$$

$$\cancel{= w * \left[\frac{1}{2} * \frac{2}{5} * 10 - \frac{1}{2} *\right]}$$

$$\frac{\frac{2}{5}}{x} = \frac{10}{10-4}$$

$$x = \frac{\cancel{12}\,6}{\cancel{50}}$$

$$= 40 * \frac{1}{2}\left(\frac{2}{5} + \frac{1}{3}\right) * 4$$

$$= -48 \text{ KN.}$$

For max. +ve S.F. at 'C'; tail of UDL should be at just right of 'C'.

$$\therefore (S.F._{C\,max})_{+ve} = w * \text{area of I.L.D. covered by UDL}$$

$$= 40 * \frac{1}{2}\left(\cancel{\frac{12}{20}}\ \frac{3}{5} + \frac{8}{20}\right) * 4$$

$$= 80 \text{ KN}$$

For max. B.M. at 'C'; U.D.L. should be placed such that

$$\frac{z}{l} = \frac{x}{d}$$

$$\Rightarrow \frac{z}{l} = \frac{x}{d}$$

$$\Rightarrow x = \frac{z}{l} \cdot d$$

$$\Rightarrow x = \frac{2}{5} * 4$$

$$\Rightarrow x = 8/5 = 1.6 \text{ m.}$$

$$B.M._{C\,max} = w * \text{area of ILD covered by UDL}$$

$$\cancel{= 40 * 2 * \frac{1}{2}\left(3.84 + \frac{24}{5}\right)}$$

$$\frac{\frac{24}{5}}{8} = \frac{y_1}{8-1.6}$$

$$\Rightarrow y_1 = 3.84 = y_2$$

$$= 40 * \left[\frac{1}{2}\left(3.84 + \frac{24}{5}\right) * 1.6 + \frac{1}{2}\left(\frac{24}{5} + 3.84\right) * 2.4\right]$$

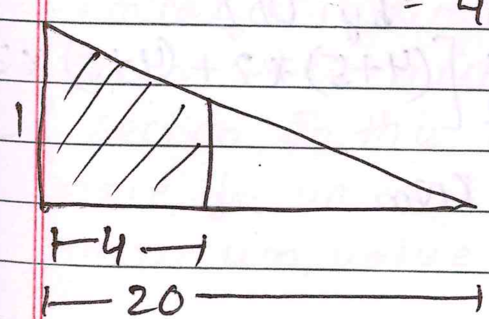
$$= 40(6.912 + 10.368)$$

$$= 691.2 \text{ KNm.}$$

Absolute max. +ve shear occurs at 'A' when tail of UDL is at 'A' and it's value is = w * area of ILD covered by UDL

$$= 40 * \left[\frac{1}{2}(1 + 0.8) * 4\right]$$

$$= 144 \text{ KN}$$



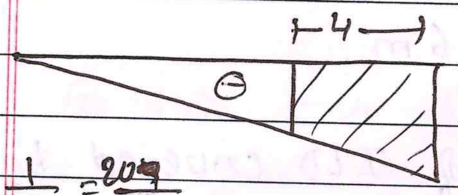
$$\frac{1}{x} = \frac{20}{16}$$

$$\Rightarrow x = 0.8$$

Absolute max. -ve shear occurs at 'B' when head of UDL is at 'B' which is = w * area of ILD covered by UDL

$$= 40 * \left[\frac{1}{2}(1+0.8) * 4\right]$$

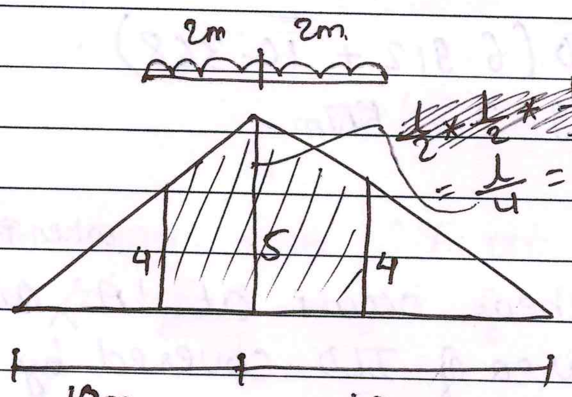
$$= -144 \text{ kN.}$$



$\frac{1}{x} = \frac{20}{16}$

$\Rightarrow x = 0.8$

Absolute max. B.M. occurs at mid-span when ~~Load is~~ mid-point of UDL is at mid-span which is,



$\frac{z(l-z)}{l}$ Here; $z = l/2$

$\frac{l/2(l - l/2)}{l}$

$= \frac{l^2/4}{l}$

$= \frac{l}{4} = \frac{20}{4} = 5$

$= w *$ area of ILD covered by UDL

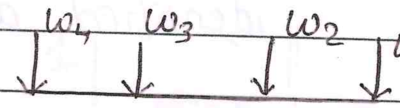
$$= 40 * \frac{1}{2}\left[(4+5)*2 + (4+5)*2\right]$$

$$= 720 \text{ kNm}$$

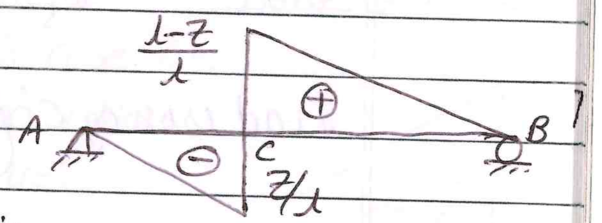
Ⓓ Train of concentrated loads:

ⓐ Maximum S.F. at any section:

Let us consider a train of concentrated load moving from left to right where W_1 is a leading load as shown in figure.



⁂ Maximum negative shear force at I.L.D. for S.F. at 'C'. 'C' occurs when most of the load is at left of 'C'. When W_1 reaches the section 'C'; -ve S.F. at 'C' is high but when it crossed the section 'C'; S.F. will decrease and again S.F. increases when 'W_2' reaches the section. So that; maximum -ve S.F. at any section occurs when one of the load is at that section. So; various nos of trials for a load at section is done and maximum S.F. is calculated.



⁂ Similarly; maximum positive shear force at 'C' will occur when most of the loads are on right of section. In this case also; various no. of trials are done for various loads crossing the section, then maximum value is taken.

(b) Maximum B.M. at any section:

To calculate maximum B.M. at any section; it is necessary to obtain the critical load over a section. Critical load is such load which changes heavier portion into lighter portion and lighter into heavier which is identified as in table below:

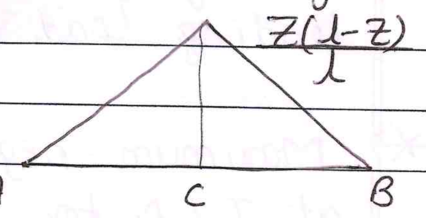
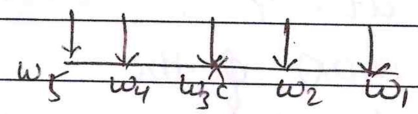


Table:

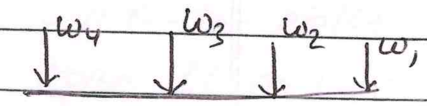


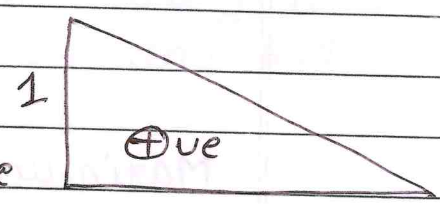
Load crossing 'C'	$\left(\frac{\text{Load}}{\text{span}}\right)$ on left (x_1)	$\left(\frac{\text{Load}}{\text{span}}\right)$ on right (x_2)	Result	
w_1	$\frac{w_2+w_3+w_4+w_5}{Z}$	$\frac{w_1}{l-Z}$	$x_1 > x_2$	Left Heavier
w_2	$\frac{w_3+w_4+w_5}{Z}$	$\frac{w_1+w_2}{l-Z}$	$x_1 > x_2$	
w_3	$\frac{w_4+w_5}{Z}$	$\frac{w_1+w_2+w_3}{l-Z}$	$x_1 < x_2$	Right Heavier

Here; critical load is at 'c'. B.M. is maximum when w_3 is at 'c'.

(c) Absolute maximum S.F. anywhere in the beam:

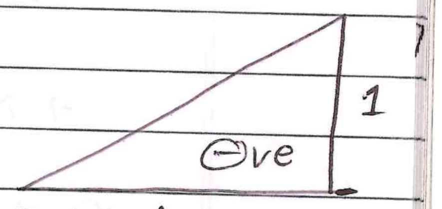
Absolute maximum positive shear at 'A' occurs when one of the load is at support 'A'. Few number of trials may be necessary to get maximum positive S.F.





I.L.D. for S.F.A.

Absolute maximum -ve shear force occurs at 'B' when one of the load is at support 'B'. In this case also; few number of trials should be done to get maximum value.



I.L.D. for S.F.B

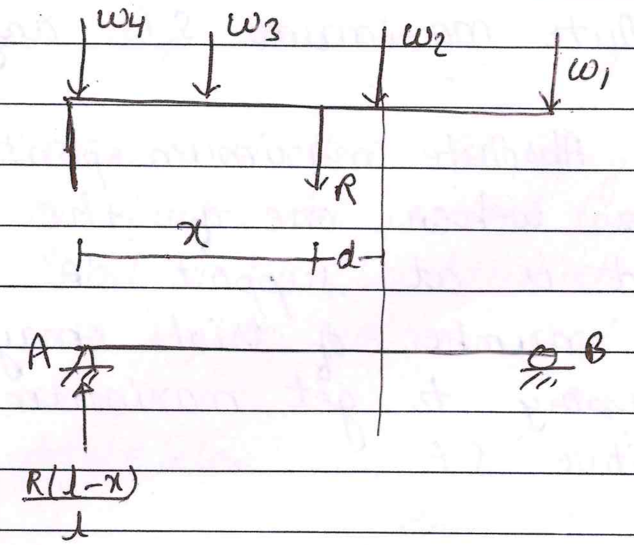
(d) Absolute maximum B.M. anywhere in the beam:

Maximum moment under a load

Let us consider train of concentrated loads moving from left to right. Let 'R' = resultant of all load. Now; the condition for maximum moment under 'w_2' is required

x = distance between resultant and 'A'

d = distance between 'R' and 'w_2'



Maximum B.M. under w_2 is:

$$M = \frac{R(l-x)(x+d)}{l} - R.d$$

$$\Rightarrow M = \frac{R}{l}(lx + ld - x^2 - xd) - Rd$$

For 'M' to be max;

$$\frac{dM}{dx} = 0.$$

$$\Rightarrow 0 = \frac{R}{l}\left(l - 2x - d\right)$$

$$\Rightarrow \boxed{x = \frac{l-d}{2}}$$

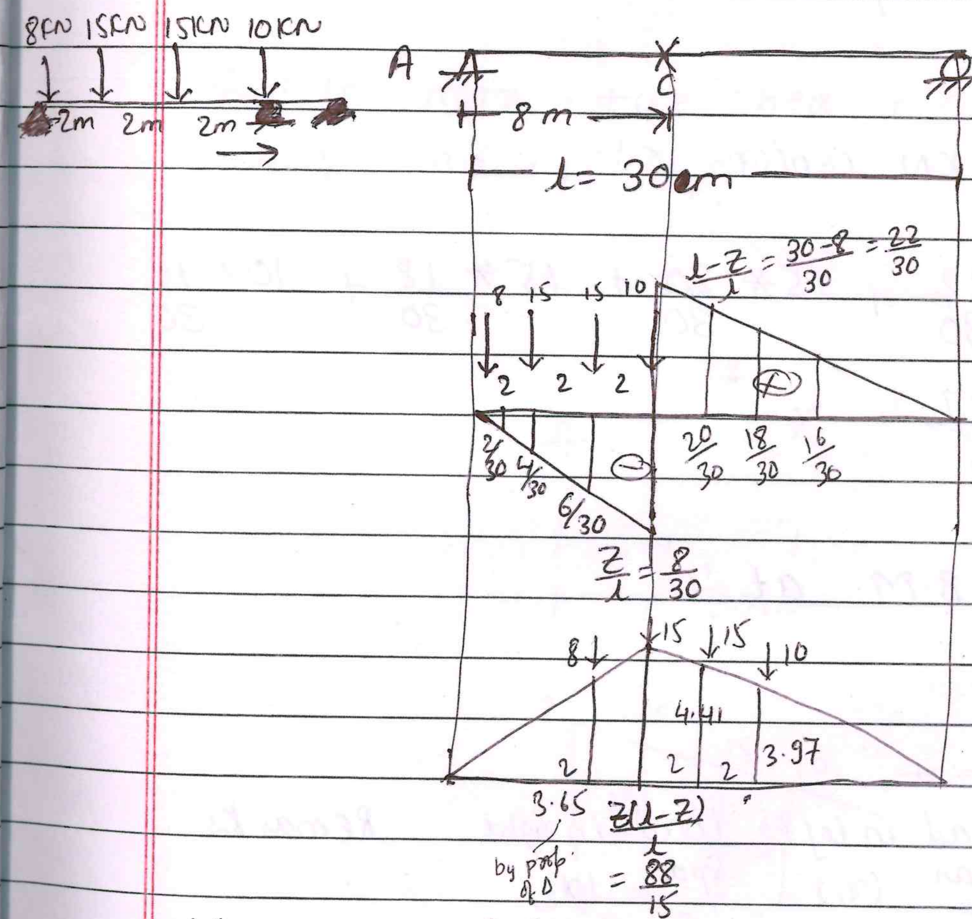
$$\Rightarrow x = \frac{l}{2} - \frac{d}{2}$$

$\therefore$ Distance of w_2 from 'A' $= x+d = \frac{l}{2} - \frac{d}{2} + d$

$$= \frac{l}{2} + \frac{d}{2}$$

From this expression it can be concluded that absolute maximum B.M. occurs at one of the load and that load and resultant should be at equidistant from center of the beam

Q. Calculate the S.F. and B.M. at 'C' if 4 point loads at 8, 15, 15, 10 kN having center-to-center spacing 2m travels a gudder of span 30 m from left-to-right having 10KN load as leading load. Also calculate absolute maximum S.F. and B.M. 'C' is 8m from left end.



Maximum $S.F._C$:

Maximum -ve S.F. at 'C' occurs when any one of the load is at 'C' (maximum load at AC span).

Trial-1:
when 10 KN is at 'C':

$$S.F._C = -10*\frac{8}{30} - 15*\frac{6}{30} - 15*\frac{4}{30} - 8*\frac{2}{30}$$
$$= -\frac{41}{5}$$

Maximum +ve S.F. occurs when maximum load is at span CB.

Trial-1:
when 8 KN crosses 'C':

$$S.F._C = 8*\frac{22}{30} + 15*\frac{20}{30} + 15*\frac{18}{30} + 10*\frac{16}{30}$$
$$= \frac{151}{5}$$

Maximum B.M. at 'C':

Table:
~~Load Crossing~~

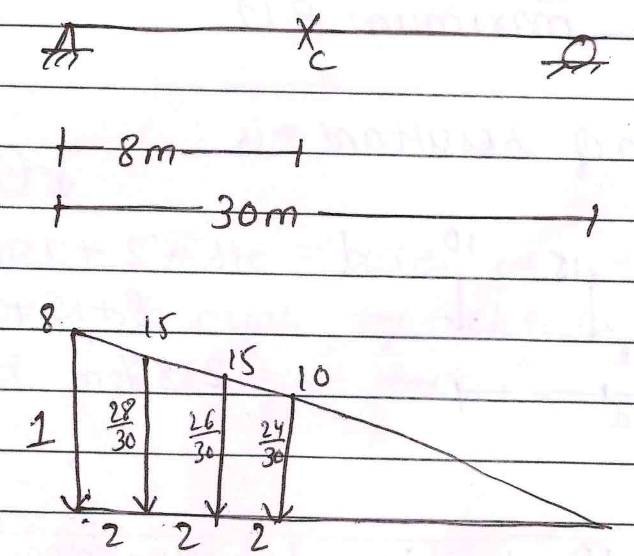
Load crossing C	Load in left span (x_1)	Load in right span (x_2)	Remarks
10	$\frac{8+15+15}{8} = 4.75$	$\frac{10}{22} = 0.45$	$x_1 > x_2$
15 (I)	$\frac{15+8}{8} = 2.875$	$\frac{15+10}{22} = 1.136$	$x_1 > x_2$
15 (II)	$\frac{8}{8} = 1$	$\frac{15+15+10}{22} = 1.818$	$x_2 > x_1$

Here; 15(II) changes heavier left span to lighter.
So; 15(II) is critical load.

∴ For maximum B.M. at 'C'; 15 KN (II) should be at 'C'.

$$\text{Max. B.M. at 'C'} = 8*3.65 + 15*\frac{88}{15} + 15*4.41 + 10*3.97$$
$$= 207.95 \text{ KNm}$$

Absolute max. +ve shear occurs at 'A' when any of load is over 'A'.



$$\frac{1}{30} = \frac{x}{28}$$

Trial 1:
S.F. when 8 KN is at 'A';

$$S.F. = 8*1 + 15*\frac{28}{30} + 15*\frac{26}{30} + 10*\frac{24}{30} = 43$$

Trial 2:
when 15(I) KN is at 'A';

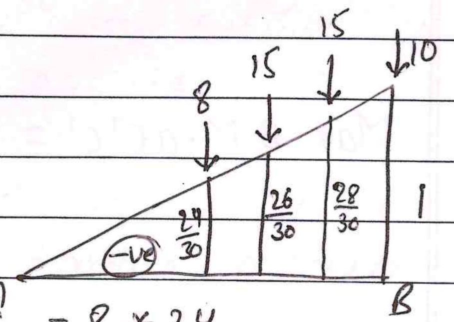
∴ $S.F._{max} = 43$ KN at 'A'

S.F. = 15*1 + 15*28 ...

Absolute maximum -ve shear occurs at 'B' when any of the load is over 'B'.

Trial #1:

when 10 KN is above B:



$$S \cdot F_B = -10*1 - 15*\frac{28}{30} - 15*\frac{26}{30} - 8*\frac{24}{30}$$

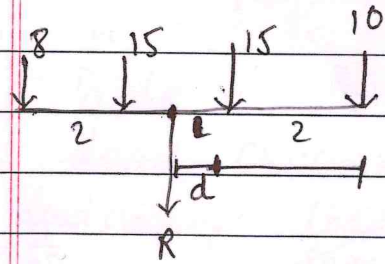
$$= -43.4 \text{ KN}$$ (-ve)

Can do trial 2 as well to check.

Absolute maximum B.M:

Position of resultant is:

(Taking moment under 10 KN) can take at any pt.



$$d = \frac{15*2 + 15*4 + 8*6}{8+15+15+10}$$

$$= 2.87 \text{ m}$$

∴ Resultant lies ~~between~~ near 15 KN (I) moment occur under 15 KN (I).

$$d = 2.87 - 2 = 0.87 \quad (\text{between } 15(I) \text{ \& } R)$$

$$\frac{d}{2} = \frac{0.87}{2} = 0.437$$

$$Z = 15 + 0.437 = 15.437 \text{ m}$$

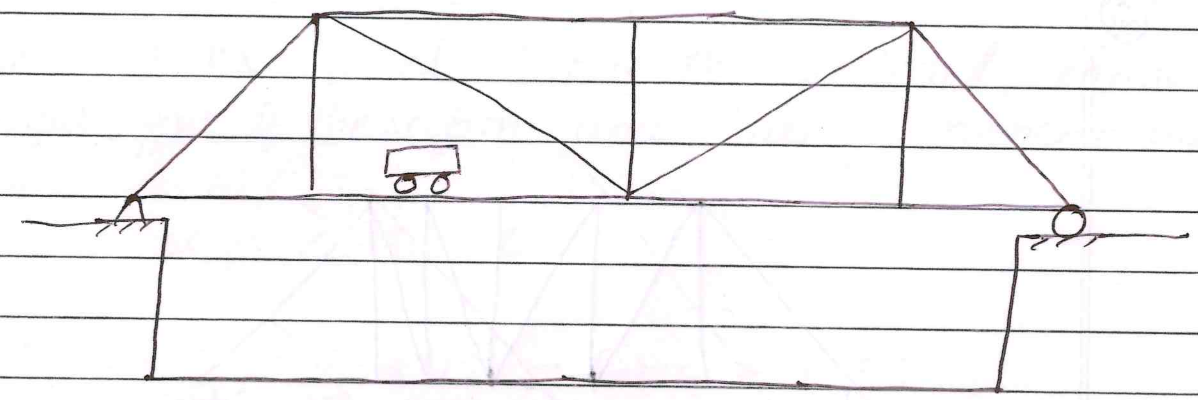
$$L - Z = 30 - 15.437 = 14.563 \text{ m}$$

ILD for bridge trusses:

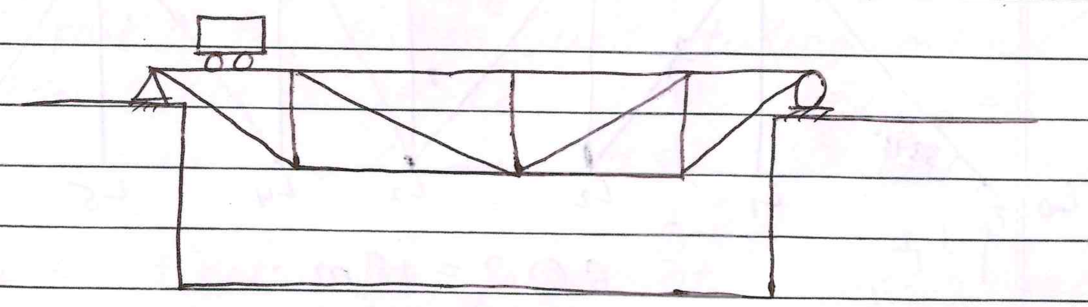
2 types:

1. Through type truss
2. Deck type truss

(1) Through type truss: Those truss in which loads / vehicles travels through bottom chord members are called through type truss.

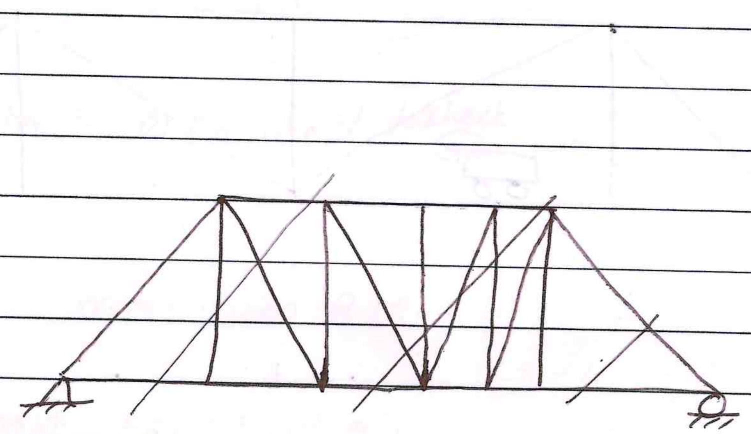


(2) Deck type truss: Those trusses in which load / vehicle moves through top chord members are called deck type trusses.

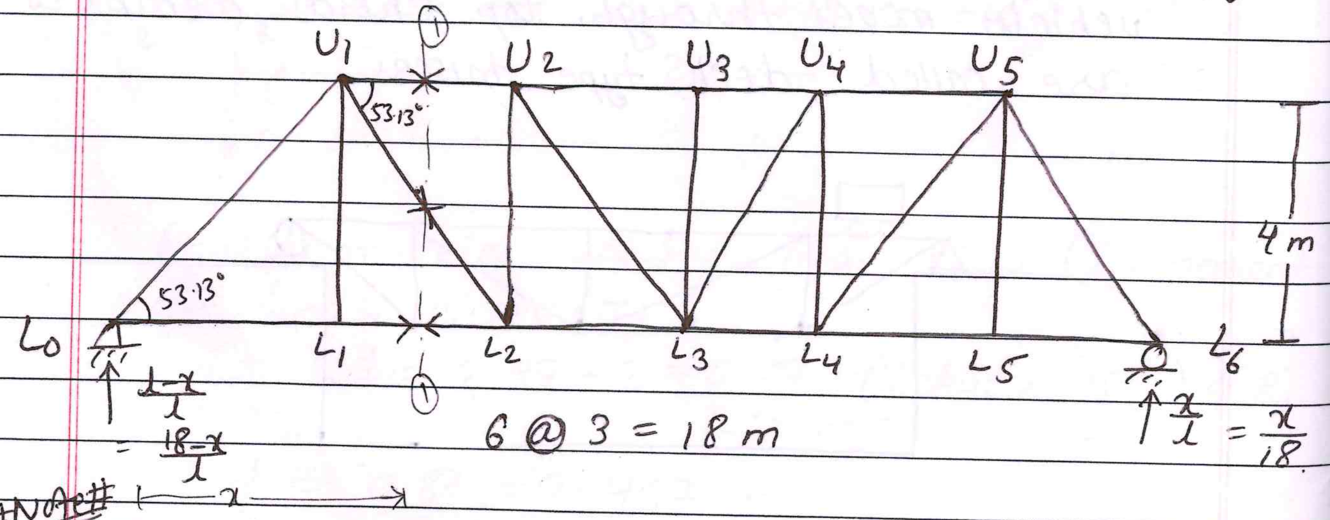


In case of bridge trusses; the load through pannel is transferred to the cross gridder. Joints of trusses are connected to cross ~~girder~~ + girder. Hence; the load on grider is transmitted to truss. As truss is a pin connected member; the upcoming load is resisted by axial compression or axial tension. As the value of axial force for a member varies as load travels; it is necessary to plot ILD for member forces in truss.

~~Q~~



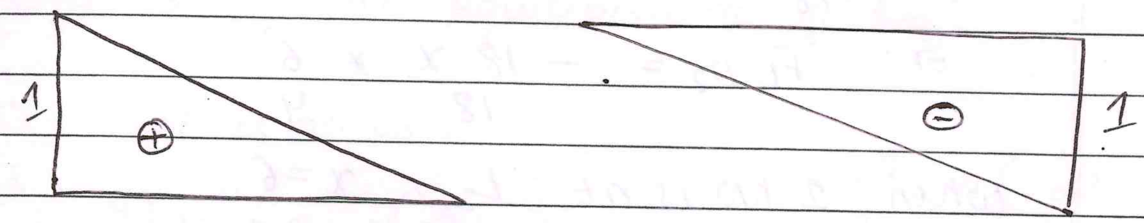
Q. Plot ILD for through type truss as shown in figure.



6 @ 3 = 18 m

#Note# ILD for reaction is same as in case of beam.

~~Top chord members:~~



ILD for RA　　　　ILD for RB

Top chord members:

I.L.D. for U_1U_2:

Consider section ①-①:

When 1 KN is at left of the section, considering right part of the section and taking moment about L_2;

for easyness as we shouldnot take moment of moving vehicle.

i.e. $\sum M_{L_2} = 0$

$$\Rightarrow -\frac{x}{18} * 12 - F_{U_1U_2} * 4 = 0$$

$$\Rightarrow F_{U_1U_2} = -\frac{x}{18} * \frac{12}{4}$$

$$\Rightarrow F_{U_1U_2} = -\frac{x}{6}$$

~~When 1 KN is at right of section, considering left part of the section and taking mome~~

When 1 kN is at L_0; $x = 0$. $\therefore F_{U_1U_2} = 0$

at L_1; $x = 3$ $\therefore F_{U_1U_2} = -1/2$.

When 1 kN is at joints at right of section; considering left equilibrium ~~of and~~ at L_2;

i.e. $\sum M_{L_2} = 0$

$\Rightarrow \frac{18-x}{18} * 6 + F_{U_1U_2} * 4 = 0.$

$\Rightarrow F_{U_1U_2} = -\frac{18-x}{18} * \frac{6}{4}$

when 1 KN is at L_2; $x = 6$.

$\therefore F_{U_1U_2} = -\frac{18-6}{18} * \frac{6}{4} = -1$

when 1 KN is at L_3; $x = 9$.

$\therefore F_{U_1U_2} = -\frac{18-9}{18} * \frac{6}{4} = -\frac{3}{4}$

when 1 KN is at L_4; $x = 12$

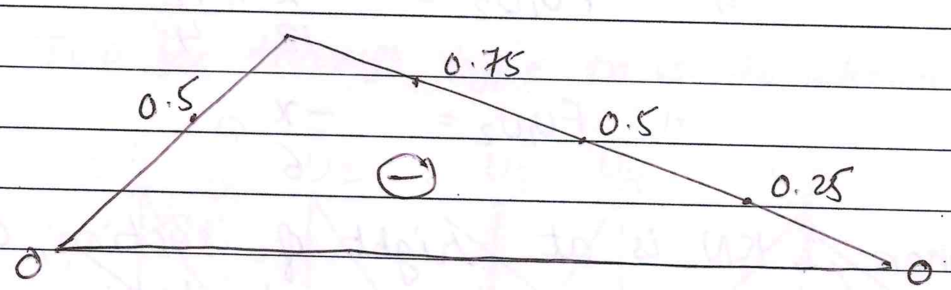
$\therefore F_{U_1U_2} = -\frac{(18-12)}{18} * \frac{6}{4} = -\frac{1}{2}$

when 1 KN is at L_5; $x = 15$

$\therefore F_{U_1U_2} = -\frac{(18-15)}{18} * \frac{6}{4} = -\frac{1}{4}$

when 1 KN is at L_6; $x = 18$.

$\therefore F_{U_1U_2} = -\frac{(18-18)}{18} * \frac{6}{4} = 0$

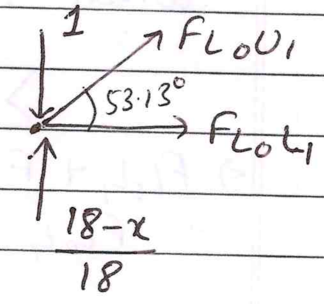


I.L.D. for L_0U_1:

Consider joint equilibirium of L_0:

when 1KN is at L_0;

$\sum V_0 = 0$ (↑+ve)

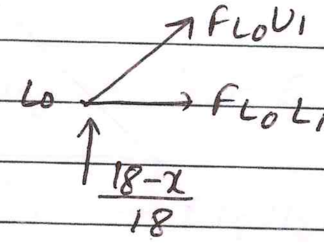


$\Rightarrow -1 + \frac{18-x}{18} + F_{L_0U_1}\sin 53.13° = 0$

$\Rightarrow F_{L_0U_1} = 0$ ($\because x = 0$)

when 1 KN is at joint other than L_0;

$\sum V_0 = 0$ (↑+ve)

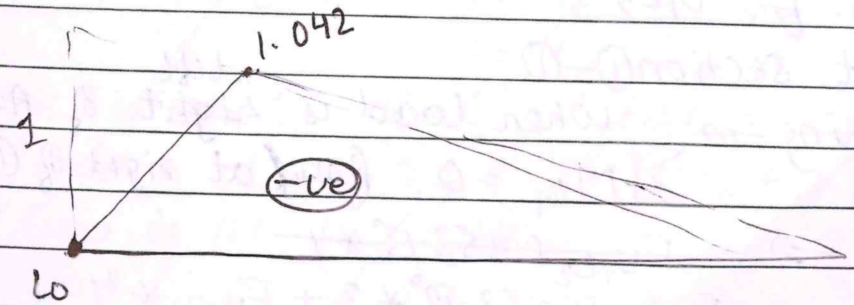


$\Rightarrow F_{L_0U_1}\sin 53.13° + \frac{18-x}{18} = 0$

$\Rightarrow \boxed{F_{L_0U_1} = -\frac{\left(\frac{18-x}{18}\right)}{\sin 53.13°}}$ Put x = ... (−)

$\Rightarrow F_{L_0U_1} = -\frac{R_1}{0.8}$ Alternative

$\therefore \boxed{\text{I.L.D. for } F_{L_0U_1} = -\frac{\text{ILD for } R_1}{0.8}}$ OR by keeping x = ... (−)



Imp

Alternative method can't be used except to top and bottom chord members as we don't know nature (+ve or -ve) of those ~~fore~~ members

But In bottom chord → tension | In top → compression

I.L.D. for L_0L_1:

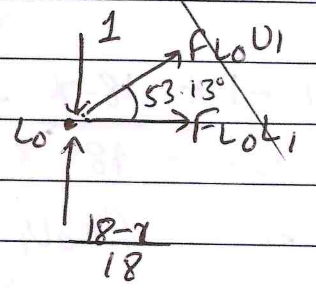
Consider joint equilibirium of L_0:

when 1kN is at L_0:

$$\sum H = 0 \quad (\rightarrow +ve)$$

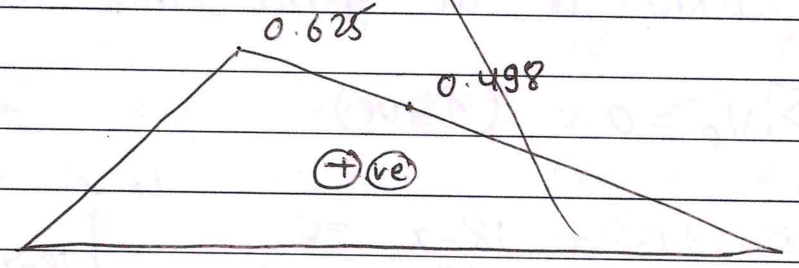
$\Rightarrow F_{L_0L_1} + F_{L_0U_1}\cos 53.13° = 0.$

$\Rightarrow F_{L_0L_1} = -F_{L_0U_1}\cos 53.13°$



~~$\Rightarrow F_{L_0L_1} =$~~

I.L.D. for L_0L_1 = −(I.L.D. for L_0U_1) * cos 53.13°



reason of cutting

I.L.D. for L_1L_2:

$$\sum H = 0 \quad (\rightarrow +ve)$$

$\Rightarrow F_{L_1L_2} - F_{L_0L_1} = 0$

$\Rightarrow F_{L_1L_2} = F_{L_0L_1}$

$\Rightarrow$ I.L.D. for L_1L_2 = I.L.D. for L_0L_1

I.L.D. for U_1L_2:

At section ①-①:

~~Taking m~~ when load is ~~right~~ left of ~~the~~ section,

$\sum M_{L_1} = 0$ (taking at right of ①-①)

~~$\Rightarrow F_{U_1L_2}\cos 53.13° * 4$~~

$\Rightarrow F_{U_1L_2}\sin 53.13° * 3 + F_{U_1U_2} * 4 + \frac{x}{18} * 15 = 0$

$\Rightarrow F_{U_1L_2}\sin 53.13° * 3 - \frac{x}{6} * 4 + \frac{x}{18} * 15 = 0$

$\Rightarrow F_{U_1L_2} = \frac{-x}{18\sin 53.13°}$

$\Rightarrow F_{U_1L_2} = \frac{-(x/18)}{\sin 53.13°}$

$\Rightarrow F_{U_1L_2} = -\frac{R_B}{\sin 53.13°}$

$\Rightarrow F_{U_1L_2} = -\frac{R_B}{0.8}$

$\Rightarrow$ I.L.D. for $U_1L_2 = -\frac{\text{I.L.D. for } R_B}{0.8}$

when load is at right of section:

$\sum M_{L_1} = 0$ (taking at left of ①-①)

should take at same pt for both case

$\Rightarrow F_{U_1L_2}\cos 53.13° * 4 + F_{U_1U_2} * 4 + \frac{18-x}{\cancel{18}\,6} * 3 = 0$

$\Rightarrow F_{U_1L_2}\cos 53.13° * 4 - \frac{x}{6} * 4 + \frac{18-x}{6} = 0$

$\Rightarrow F_{U_1L_2}\cos 53.13° * 4 = \frac{4x}{6} - \frac{18-x}{6}$

$\Rightarrow F_{U_1L_2} = \frac{5x-18}{24 * \cos 53.13°}$

$\Rightarrow F_{U_1L_2} = \frac{5x-18}{14.4}$

when load is at L_2; $x=6$ $\therefore F_{U_1L_2} = 0.83$ kN

when load is at L_3; $x=9$ $\therefore F_{U_1L_2} = 1.875$ kN

when load is at L_4; $x=12$ $\therefore F_{U_1L_2} = 2.92$ kN

when load is at L_5; $x=15$ $\therefore F_{U_1L_2} = 3.95$ kN

when load is at L_6; $x=18$ $\therefore F_{U_1L_2} = 5$ kN